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This paper is mainly concerned with the construction of a theory of material behavior with infinitesimal strain accompanied by moderate rotation. After introducing a definition for moderate rotation and establishing a number of theorems pertaining to its properties, precise estimates are obtained for the (local) moderate rotation and related kinematical results in terms of infinitesimal strain. For motions which result in small strain accompanied by moderate rotation, (continued)

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20. Abstract (continued)

the invariance of constitutive equations under arbitrary superposed rigid body motions is discussed with particular reference to an elastic material.

Small Strain Accompanied by Moderate Rotation

by P. M. Naghdi and L. Vongsarnpigoon

Abstract. This paper is mainly concerned with the construction of a theory of material behavior with infinitesimal strain accompanied by moderate rotation. After introducing a definition for moderate rotation and establishing a number of theorems pertaining to its properties, precise estimates are obtained for the (local) moderate rotation and related kinematical results in terms of infinitesimal strain. For motions which result in small strain accompanied by moderate rotation, the invariance of constitutive equations under arbitrary superposed rigid body motions is discussed with particular reference to an elastic material.

Contents

1. Introduction

In order to state clearly the purpose of this paper and to motivate the developments that follow, consider first the manner of construction of an infinitesimal theory of deformation. In constructing such a theory, if as usual it is assumed that the displacement gradient is small, then both the strain and the (local) rotation are small also. On the other hand, if only the strain tensor is assumed to be small, then the rotation tensor is not necessarily small. It is then of interest to ask whether or not any condition imposed on the strain field would suffice to ensure the smallness of the rotation also. More generally, suppose it is desired to have the rotation moderately large in some sense. Is it then possible to restrict the strain field so as to ensure that it is accompanied by moderate rotation?

Some insight into the above questions is provided by a well-known result of the infinitesimal theory which states that the gradient of the infinitesimal rotation can be expressed in terms of the gradient of the strain field (see for example Sokolnikoff, 1956, p. 27). It is not difficult to see that some corresponding result should exist even when the deformation is not infinitesimal. Indeed, in the context of a finitely deformed body, the deformation gradient tensor $\underset{\sim}{F}$ can be expressed as a product of the (local) rotation tensor $\underset{\sim}{R}$ and the stretch tensor $\underset{\sim}{U}$ which also determines a measure of strain such as the relative Lagrangian strain $\underset{\sim}{E}$. That there must exist some connection between the gradient of the stretch $\underset{\sim}{U}$ (or the gradient of the strain $\underset{\sim}{E}$) and the gradient of the rotation $\underset{\sim}{R}$ becomes evident when one recalls the compatibility condition that $\underset{\sim}{F}$ must satisfy. The main purpose of the present paper is to derive a representation for the rotation and deformation field directly from the strain, which

can then be used to estimate the magnitude of $\underset{\sim}{R}$ in terms of the magnitude of $\underset{\sim}{U}$ or $\underset{\sim}{E}$. Results of this kind are of interest in various contexts and especially for materials undergoing deformation in which the strain and rotation are not necessarily of the same orders of magnitude. In fact, in a number of theories for special bodies, such as those for shells and rods, there are circumstances in which the motion results in small strain, while the rotation and the deformation may be large or <u>moderately large</u>.

Motivated in part by the remarks made in the preceding paragraph, most of the paper is concerned with the development of a procedure for the construction of a theory of infinitesimal strain accompanied by moderate rotation, although the procedure is applicable to other situations in which the strain and the rotation may be of different orders. Thus, after introducing a geometrically appealing definition for moderate rotation and establishing a number of relevant theorems pertaining to its properties, estimates are obtained for the angle of rotation and the moderate rotation tensor and these are eventually expressed in terms of infinitesimal strain. These and related kinematical results are then put in a properly invariant form, and the invariance of constitutive equations under arbitrary superposed rigid body motions is discussed in the case of an elastic material undergoing small strain accompanied by moderate rotation. Although throughout the paper use is made of the direct (coordinate-free) notation, which often allows the results to be stated in their simplest form, on occasions we also employ the component forms (at least partially) of the various equations since these are more convenient in explicit calculations. A brief account of the notations used and some mathematical preliminaries is given at the end of this section and additional notations and mathematical terminology are collected in Appendix A.

The problem of the determination of the rotation tensor $\underset{\sim}{R}$ from a strain tensor, and the connection between them, has a long history that dates back o Cauchy. A detailed account of the subject can be found in sections 34-38 and 55-57 of Truesdell and Toupin (1960). Truesdell and Toupin (1960, p. 276) also discuss in some detail a measure of rotation -- called mean rotation -- introduced by Novozhilov (1953) and use this in their development of a theory of infinitesimal strain and infinitesimal rotation (Truesdell and Toupin 1960, p. 305). More recently, in the context of finite deformation, the problem of the determination of the rotation tensor $\underset{\sim}{R}$ from the strain tensor has been discussed by John (1961) and by Shield (1973). In particular, Shield has derived a useful representation of the integrability relations for finite strain in a Euclidean space, and has further shown that for two-dimensional deformation the rotation and deformation field can be determined directly from the knowledge of strain by a line integral.

1.1 Scope and outline of contents

After collecting some kinematical and kinetical results in section 2, for clarity and ease of reference we include a sketch of the derivation of the differential equations for rotation $\underset{\sim}{R}$ in section 3. Since the presentation of this development differs somewhat from that given by Shield (1973), one or two of the details are collected in the first part of Appendix B (between (B1) and (B4)) where a correspondence with Shield's (1973) main results are indicated. We also take this opportunity to include in section 3 two elementary kinematical theorems concerning the rotation $\underset{\sim}{R}$ and the deformation gradient which are utilized later in the paper.

We begin section 4 by defining in precise terms four measures of smallness denoted by $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and associated, respectively, with

strain, strain gradient, rotation and rotation gradient [see the expressions (4.1)-(4.2) and (4.5)-(4.6)]. We then observe the conditions under which these four measures can be related [see (4.7)] and go on to show that the rotation gradient is of the same order of magnitude as the strain gradient, i.e., ε_3 is of the same order as ε_1 [see (4.17)]. Under the additional assumption that the rotation tensor is the identity tensor at some material point in the body, we prove in subsection 4.1 that the rotation tensor is of the same order of magnitude as the strain gradient, i.e., ε_2 is of the same order as ε_1 [see (4.26)]. Later, making use of a further assumption that relates the two measures of smallness ε_0 and ε_1, we express the rotation tensor and all other kinematical quantities in terms of ε_0 or equivalently the infinitesimal strain.

In the second part of section 4, we introduce two separate definitions: One for (1) infinitesimal strain accompanied by infinitesimal rotation (Definition 4.1) and another (2) for infinitesimal strain accompanied by moderate rotation (see Definition 4.2). Although our main objective here is to deal with the latter, we find it instructive to prove a number of results in subsection 4.2 pertaining to the infinitesimal theory of motion. The remainder of section 4 (subsections 4.3 and 4.4) deals entirely with the case of infinitesimal strain accompanied by moderate rotation. In Theorem 4.3, we essentially prove that if the strain is of $\underset{\sim}{O}(\varepsilon_0)$ and $\underset{\sim}{R}$ is a moderate rotation, then the relative displacement gradient is of $\underset{\sim}{O}(\varepsilon_0^{1/2})$. In a subsequent Theorem 4.4, with the help of a known representation for $\underset{\sim}{R}$ in terms of angle of rotation θ and the results stated in the preceding paragraph, we estimate the order of magnitude of $\underset{\sim}{R}$ in terms of the infinitesimal strain We also obtain in subsection 4.3 expressions for the deformation gradient $\underset{\sim}{F}$ and the relative displacement gradient $\underset{\sim}{H}$ when

the motion is such that the strain is of $O(\varepsilon_0)$ but is accompanied by moderate rotation of $O(\varepsilon_0^{1/2})$.

In a recent paper, Casey and Naghdi (1981) have constructed a properly invariant theory in which, apart from superposed rigid body motions, both the strain and rotation (and hence also the relative displacement gradient) are infinitesimal and have further shown that in this infinitesimal theory of motion the constitutive equations, as well as the equations of motion, are all properly invariant under arbitrary (not necessarily infinitesimal) superposed rigid body motions. In the latter part of section 4 (see subsection 4.4), by using the procedure of Casey and Naghdi (1981) we remove from all motions the translation and rotation at any particle Y of the body, thereby rendering all kinematical quantities (including the infinitesimal strain) properly invariant under superposed rigid body motions. Finally, in section 5, in a manner similar to that of Casey and Naghdi (1981) we briefly discuss the invariance properties of the constitutive equations for small strain accompanied by moderate rotation and further demonstrate in Appendix C that, as in the case of the infinitesimal theory, for the theory under discussion the particular choice of the particle Y is again immaterial in effecting the properly invariant nature of the results.

1.2 Notation and mathematical preliminaries

We close this section with a short glossary of notations and some mathematical terminology that will be needed in what follows. Any linear mapping from V, the three-dimensional translation vector space associated with the Euclidean point space $\mathcal{E}$, into V will be called a second order tensor. The trace and determinant functions of second order tensors will be denoted, respectively, by tr and det. The transpose of a second order

tensor $\underset{\sim}{T}$ will be denoted by $\underset{\sim}{T}^T$, while the inverse of $\underset{\sim}{T}$ if it exists will be denoted by $\underset{\sim}{T}^{-1}$. The usual inner product on V is written $\underset{\sim}{a} \cdot \underset{\sim}{b}$ for any two vectors $\underset{\sim}{a},\underset{\sim}{b} \in V$ and the (induced) norm, or magnitude, of $\underset{\sim}{a}$ is given by $\|\underset{\sim}{a}\| = (\underset{\sim}{a} \cdot \underset{\sim}{a})^{\frac{1}{2}}$. The set of second order tensors can be provided with an inner product $\underset{\sim}{A} \cdot \underset{\sim}{B} = \mathrm{tr}(\underset{\sim}{A}^T\underset{\sim}{B})$ and a norm $\|\underset{\sim}{A}\| = (\underset{\sim}{A} \cdot \underset{\sim}{A})^{\frac{1}{2}}$ for any second order tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$. We note that the definition of the norm of a second order tensor satisfies the usual properties of the norm, i.e.,

$$\|\underset{\sim}{A}\| \geq 0 \quad , \quad \text{and} \quad \|\underset{\sim}{A}\| = 0 \ \text{ if and only if } \ \underset{\sim}{A} = \underset{\sim}{0} \quad ,$$

$$\|\alpha\underset{\sim}{A}\| = |\alpha| \, \|\underset{\sim}{A}\| \quad , \tag{1.1}$$

$$\|\underset{\sim}{A} + \underset{\sim}{B}\| \leq \|\underset{\sim}{A}\| + \|\underset{\sim}{B}\|$$

for any second order tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ and for any scalar α. Further, it can be shown that

$$\|\underset{\sim}{A}\,\underset{\sim}{a}\| \leq \|\underset{\sim}{A}\|\,\|\underset{\sim}{a}\| \tag{1.2}$$

for any second order tensor $\underset{\sim}{A}$ and any vector $\underset{\sim}{a}$. In addition, for any second order tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$, we have

$$\|\underset{\sim}{A}\,\underset{\sim}{B}\| \leq \|\underset{\sim}{A}\|\,\|\underset{\sim}{B}\| \quad . \tag{1.3}$$

The tensor product $\underset{\sim}{a} \otimes \underset{\sim}{b}$ of any two vectors $\underset{\sim}{a},\underset{\sim}{b} \in V$ is the second order tensor defined by $(\underset{\sim}{a} \otimes \underset{\sim}{b})\underset{\sim}{v} = (\underset{\sim}{b} \cdot \underset{\sim}{v})\underset{\sim}{a}$ for every vector $\underset{\sim}{v}$. We recall the formulae $\mathrm{tr}(\underset{\sim}{a} \otimes \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{b}$, $(\underset{\sim}{a} \otimes \underset{\sim}{b})^T = \underset{\sim}{b} \otimes \underset{\sim}{a}$, $(\underset{\sim}{a} \otimes \underset{\sim}{b})(\underset{\sim}{c} \otimes \underset{\sim}{d}) = (\underset{\sim}{a} \otimes \underset{\sim}{d})(\underset{\sim}{b} \cdot \underset{\sim}{c})$, $(\underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot (\underset{\sim}{c} \otimes \underset{\sim}{d}) = \underset{\sim}{a} \cdot \underset{\sim}{c}\ \underset{\sim}{b} \cdot d$, which hold for all vectors $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{d} \in V$. In addition, we note that

$$\|\underset{\sim}{a} \otimes \underset{\sim}{b}\| = \|\underset{\sim}{a}\|\,\|\underset{\sim}{b}\| \tag{1.4}$$

for all $\underset{\sim}{a},\underset{\sim}{b} \in V$.

2. Some preliminary kinematical and kinetical results

Let X be a particle of a body $\mathcal{B}$ and denote by $\underset{\sim}{X}$ and $\underset{\sim}{x}$ the position vectors of X in a fixed reference configuration $\underset{\sim}{\kappa}_o$ and in the current configuration $\underset{\sim}{\kappa}$, respectively. A motion of $\mathcal{B}$ is the mapping $\underset{\sim}{\chi}$ which assigns the position vector $\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X},t)$ to the particle X at time t. In the configuration $\underset{\sim}{\kappa}_o$, let $\mathcal{B}$ occupy a region* $\mathcal{R}_o$ embedded in a 3-dimensional Euclidean space $\mathcal{E}$ and denote the boundary of $\mathcal{R}_o$ by $\partial\mathcal{R}_o$. The image of $\mathcal{R}_o$ in $\underset{\sim}{\kappa}$ will be denoted by $\mathcal{R}$ with boundary $\partial\mathcal{R}$. We assume that at each fixed t, the mapping $\underset{\sim}{\chi}$ of $\mathcal{R}_o$ into $\mathcal{R}$ possesses a smooth inverse denoted by $\underset{\sim}{\chi}^{-1}$.

Let† $\underset{\sim}{F} = \text{Grad}\,\underset{\sim}{\chi}$ be the deformation gradient relative to the configuration $\underset{\sim}{\kappa}_o$ and recall that $\det \underset{\sim}{F} > 0$. Then, by the polar decomposition theorem, $\underset{\sim}{F}$ can be decomposed in the form

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} \quad , \tag{2.1}$$

where the right stretch $\underset{\sim}{U}$ is a symmetric positive definite second order tensor, the (local) rotation $\underset{\sim}{R}$ is a proper orthogonal tensor satisfying

$$\underset{\sim}{R}^T\underset{\sim}{R} = \underset{\sim}{R}\,\underset{\sim}{R}^T = \underset{\sim}{I} \quad , \quad \det \underset{\sim}{R} = 1 \tag{2.2}$$

and $\underset{\sim}{I}$ stands for the identity tensor. The right Cauchy-Green measure of deformation $\underset{\sim}{C}$ can be represented in the forms

$$\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F} = \underset{\sim}{U}^2 \tag{2.3}$$

and the relative (Lagrangian) finite strain tensor $\underset{\sim}{E}$ is given by

$$\underset{\sim}{E} = \tfrac{1}{2}(\underset{\sim}{C}-\underset{\sim}{I}) \quad . \tag{2.4}$$

---

*A region is regarded here as a nonempty connected and compact subset of $\mathcal{E}$ having a piecewise smooth boundary.

†Definition of the gradient of a vector function is given by $(A4)_1$ of Appendix A.

In order to express certain expressions in component form, it is convenient to employ two fixed right-handed orthonormal bases $\{\underset{\sim}{e}_A\}$, $(A = 1,2,3)$, and $\{\underset{\sim}{e}_i\}$, $(i = 1,2,3)$, in V, the former basis being used for vector fields defined on the region ${}_0\mathcal{R}$ and the latter for vector fields defined on $\mathcal{R}$. The convention of summation over repeated Latin index will be employed. Thus, for example, we write $\underset{\sim}{X} = X_A \underset{\sim}{e}_A$, and $\underset{\sim}{x} = x_i \underset{\sim}{e}_i$. Furthermore, a second order tensor $\underset{\sim}{A}$ may be represented by $A_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$, $A_{iM} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_M$, or $A_{MN} \underset{\sim}{e}_M \otimes \underset{\sim}{e}_N$ as appropriate, where $A_{ij} = \underset{\sim}{e}_i \cdot \underset{\sim}{A}\, \underset{\sim}{e}_j = \underset{\sim}{A} \cdot (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)$, etc.

The relative displacement $\underset{\sim}{u}$ and the relative deformation gradient $\underset{\sim}{H}$ defined by

$$\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X} \quad , \tag{2.5}$$

and

$$\underset{\sim}{H} = \underset{\sim}{F} - \underset{\sim}{I} = \text{Grad}\, \underset{\sim}{u} = \underset{\sim}{u}_{,K} \otimes \underset{\sim}{e}_K \quad , \tag{2.6}$$

where a comma stands for partial differentiation with respect to the coordinate X_K. The strain tensor $\underset{\sim}{E}$ can be expressed as

$$\begin{aligned} \underset{\sim}{E} &= \tfrac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T + \underset{\sim}{H}^T \underset{\sim}{H}) \\ &= \tfrac{1}{2}(u_{L,K} + u_{K,L} + u_{M,K} u_{M,L}) \underset{\sim}{e}_L \otimes \underset{\sim}{e}_K \quad . \end{aligned} \tag{2.7}$$

Also, it is easily seen from (2.4) that Grad $\underset{\sim}{C} = 2$ Grad $\underset{\sim}{E}$ or equivalently with the use of $(A5)_2$,

$$\underset{\sim}{C}_{,K} = 2\underset{\sim}{E}_{,K} = (\underset{\sim}{U}\underset{\sim}{U})_{,K} = \underset{\sim}{U}\underset{\sim}{U}_{,K} + \underset{\sim}{U}_{,K}\underset{\sim}{U} \quad . \tag{2.8}$$

A motion $\underset{\sim}{\chi}^+$ is said to differ from $\underset{\sim}{\chi}$ by a superposed rigid body motion if and only if

$$\underset{\sim}{\chi}^+(\underset{\sim}{X}, t^+) = \underset{\sim}{Q}(t)\underset{\sim}{\chi}(\underset{\sim}{X}, t) + \underset{\sim}{a}(t) \quad , \quad t^+ = t + a \tag{2.9}$$

for some proper orthogonal second order tensor-valued function $\mathbf{Q}(t)$ of time, some vector-valued function $\mathbf{a}(t)$ of time and some real constant a. The configuration of $\mathscr{B}$, at time t^+ in the motion $\boldsymbol{\chi}^+$ will be denoted by $\boldsymbol{\kappa}^+$. The deformation gradient $\mathbf{F}^+(\mathbf{X},t^+)$ calculated from (2.9) is related to $\mathbf{F}$ by

$$\mathbf{F}^+ = \mathbf{Q}(t)\mathbf{F} \quad . \tag{2.10}$$

Then, it can be easily verified from (2.3), (2.4) and (2.1) that

$$\mathbf{C}^+ = \mathbf{C} \ , \ \mathbf{U}^+ = \mathbf{U} \ , \ \mathbf{E}^+ = \mathbf{E} \ , \ \mathbf{E}^+_{,K} = \mathbf{E}_{,K} \ , \ \mathbf{R}^+ = \mathbf{Q}(t)\mathbf{R} \quad . \tag{2.11}$$

The relative displacement field associated with $\boldsymbol{\chi}^+$ is $\mathbf{u}^+ = \mathbf{x}^+ - \mathbf{X}$ and its gradient $\mathbf{H}^+ = \text{Grad } \mathbf{u}^+$ in terms of (2.6) is given by

$$\mathbf{H}^+ = \mathbf{Q}\mathbf{F} - \mathbf{I} = \mathbf{Q}\mathbf{H} + \mathbf{Q} - \mathbf{I} \quad . \tag{2.12}$$

Thus, $\mathbf{C},\mathbf{U},\mathbf{E}$ are unaltered under superposed rigid body motions, while $\mathbf{R}$ is unaltered apart from orientation[§]. The displacement gradient $\mathbf{H}$, however, is neither unaltered nor unaltered apart from orientation under the transformation (2.9).

For later reference, we recall here an exact representation for the rotation tensor $\mathbf{R}$, namely[*]

$$\mathbf{R} = \mathbf{I} + \boldsymbol{\Phi} + \boldsymbol{\Psi} \quad , \tag{2.13}$$

where

[§]For the motivation and the precise meaning of this terminology, see Green and Naghdi (1979).

[*]See Truesdell and Toupin (1960, p. 2[illegible]0, Eq. (37.17)).

$$\underset{\sim}{\Phi} = \underset{\sim}{\Phi}^T = (1 - \cos\theta)(\underset{\sim}{\mu} \otimes \underset{\sim}{\mu} - \underset{\sim}{I}) = (1 - \cos\theta)(\mu_K\mu_L - \delta_{KL})\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L \quad ,$$
$$\underset{\sim}{\Psi} = -\underset{\sim}{\Psi}^T = -\sin\theta\, \underset{\sim}{\varepsilon}\, \underset{\sim}{\mu} = -\sin\theta\, \varepsilon_{KLM}\mu_M \underset{\sim}{e}_K \otimes \underset{\sim}{e}_L \quad . \tag{2.14}$$

In $(2.14)_{1,2}$, δ_{KL} are the components of the identity tensor $\underset{\sim}{I}$, $\underset{\sim}{\varepsilon} = \varepsilon_{KLM}\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L \otimes \underset{\sim}{e}_M$ is the permutation symbol in 3-space, $\underset{\sim}{\mu} = \mu_M \underset{\sim}{e}_M$ is the unit vector in the principal direction of $\underset{\sim}{R}$ that is associated with the principal value 1, and θ represents the angle of rotation calculated from the relationship

$$\operatorname{tr} \underset{\sim}{R} = 1 + 2\cos\theta \quad , \quad -\pi < \theta \leq \pi \quad . \tag{2.15}$$

We introduce now some kinetical quantities needed in the development of section 5. Thus, with reference to the motion $\underset{\sim}{\chi}$, let $\rho = \rho(\underset{\sim}{X},t)$ be the mass density in the configuration $\underset{\sim}{\kappa}$, $\underset{\sim}{n}$ the outward unit normal to the surface $\partial\mathcal{R}$, $\underset{\sim}{t} = \underset{\sim}{t}(\underset{\sim}{X},t\,;\underset{\sim}{n})$ the stress vector acting on this surface and $\underset{\sim}{T} = \underset{\sim}{T}(\underset{\sim}{X},t)$ the associated Cauchy stress tensor. The corresponding quantities in the configuration $\underset{\sim}{\kappa}^+$ will be denoted by $\rho^+, \underset{\sim}{n}^+, \underset{\sim}{t}^+$ and $\underset{\sim}{T}^+$, respectively. We recall that under the superposed rigid body motions (2.9), ρ^+ and $\underset{\sim}{n}^+$ transform according to

$$\rho^+ = \rho \quad , \quad \underset{\sim}{n}^+ = \underset{\sim}{Q}(t)\underset{\sim}{n} \tag{2.16}$$

and adopt the usual assumption that $\underset{\sim}{t}^+$ for the motion $\underset{\sim}{\chi}^+$ is related to $\underset{\sim}{t}$ by

$$\underset{\sim}{t}^+ = \underset{\sim}{Q}(t)\underset{\sim}{t} \quad . \tag{2.17}$$

It then follows from the relation $\underset{\sim}{t} = \underset{\sim}{T}\,\underset{\sim}{n}$, $(2.16)_2$ and (2.17) that

$$\underset{\sim}{T}^+ = \underset{\sim}{Q}(t)\underset{\sim}{T}\,\underset{\sim}{Q}^T(t) \quad . \tag{2.18}$$

3. Differential equations for the rotation tensor

In order to examine locally the deformation of the body $\mathscr{B}$, consider a material line element $d\underset{\sim}{X}$ at $\underset{\sim}{X}$ in the reference configuration $\underset{\sim}{\kappa}_o$. As a result of deformation, $d\underset{\sim}{X}$ is mapped into a line element $d\underset{\sim}{x} = \underset{\sim}{F}\, d\underset{\sim}{X}$ at $\underset{\sim}{x}$ in the present configuration $\underset{\sim}{\kappa}$. Since the space $\mathcal{E}$ is 3-dimensional, in order to determine the deformation of an arbitrary material line element it will suffice to consider the deformation of three mutually orthogonal line elements $d\underset{\sim}{X}_M$, namely

$$d\underset{\sim}{x}_M = \underset{\sim}{F}\, d\underset{\sim}{X}_M \quad , \quad (M = 1,2,3) \quad . \tag{3.1}$$

Let the magnitudes of $d\underset{\sim}{X}_M$ and $d\underset{\sim}{x}_M$ be denoted by dS_M and ds_M, respectively, and introduce the unit vectors $\underset{\sim}{K}_M$ in the directions of $d\underset{\sim}{X}_M$ and the unit vectors $\underset{\sim}{k}_M$ in the direction of $d\underset{\sim}{x}_M$. Hence, the three unit vectors $\underset{\sim}{K}_M$ $(M = 1,2,3)$ form an orthonormal basis which, without loss in generality, will be identified with the orthonormal basis $\{\underset{\sim}{e}_M\}$ in $\underset{\sim}{\kappa}_o$. In general, the line elements $d\underset{\sim}{X}_M$ undergo both stretch and rotation and the ratios $(ds_1/dS_1,\ ds_2/dS_2,\ ds_3/dS_3)$ denoted by λ_M $(M = 1,2,3)$ are called the stretch of the line elements. The above observations may be summarized as

$$d\underset{\sim}{X}_M = \underset{\sim}{e}_M\, dS_M \quad , \quad d\underset{\sim}{x}_M = \underset{\sim}{k}_M\, ds_M \quad , \quad \lambda_M = \frac{ds_M}{dS_M} \ \text{(no sum on M)} \quad . \tag{3.2}$$

It follows from (3.1) and $(3.2)_{1,2,3}$ that

$$d\underset{\sim}{x}_M \cdot \underset{\sim}{e}_i = F_{iM}\, dS_M = \lambda_M\, k_{Mi}\, dS_M \quad \text{(no sum on M)} \quad , \tag{3.3}$$

where k_{Mi} are the components of the unit vector $\underset{\sim}{k}_M$ referred to the orthonormal basis $\{\underset{\sim}{e}_i\}$ in the current configuration $\underset{\sim}{\kappa}$ and F_{iM} are the components of $\underset{\sim}{F}$ referred to the basis $\{\underset{\sim}{e}_i \otimes \underset{\sim}{e}_M\}$, i.e.,

$$\underset{\sim}{F} = F_{iM}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_M = \underset{\sim}{M}_M \otimes \underset{\sim}{e}_M \quad , \quad \underset{\sim}{M}_M = F_{iM}\underset{\sim}{e}_i = \underset{\sim}{F}\, \underset{\sim}{e}_M \quad . \tag{3.4}$$

In view of (3.4), the results (3.3) may also be displayed as

$$\underset{\sim}{M}_M = \lambda_M \underset{\sim}{k}_M = \partial \underset{\sim}{\chi} / \partial X_M \quad \text{(no sum on M)} \quad . \tag{3.5}$$

It should be noted that since $\underset{\sim}{F}$ is invertible, $\underset{\sim}{M}_M$ are linearly independent and form a basis. Moreover, $\underset{\sim}{M}_M$ can be interpreted as the image of the material basis $\underset{\sim}{K}_M$ (or $\underset{\sim}{e}_M$) in the configuration $\underset{\sim}{\kappa}$. It is clear that the knowledge of the images of $\underset{\sim}{K}_M$ at every point gives full information about the components $F_{iM} = \underset{\sim}{e}_i \cdot \underset{\sim}{M}_M$. Given $\underset{\sim}{F}$ (or the components F_{iM}), the necessary and sufficient conditions for the existence of a deformation function whose gradient is $\underset{\sim}{F}$ are the compatibility conditions*

$$(\text{Grad } \underset{\sim}{F})^{T_1} = \text{Grad } \underset{\sim}{F} \quad \text{or} \quad F_{iK,L} = F_{iL,K} \quad . \tag{3.6}$$

In terms of $\underset{\sim}{M}_M$, the conditions (3.6) can also be written as

$$\underset{\sim}{M}_{M,L} = \underset{\sim}{M}_{L,K} \quad . \tag{3.7}$$

Clearly the inner product of $\underset{\sim}{M}_K$ and $\underset{\sim}{M}_L$ gives the Cauchy-Green measure, i.e.,

$$C_{KL} = F_{iK}F_{iL} = F_{iK}\underset{\sim}{e}_i \cdot F_{jL}\underset{\sim}{e}_j = \underset{\sim}{M}_K \cdot \underset{\sim}{M}_L \tag{3.8}$$

and it is seen that the components of the metric tensor associated with the material basis $\underset{\sim}{M}_K$ are C_{KL}.

Given C_{KL}, it does not necessarily follow that there exists a deformation function $\underset{\sim}{\chi}$ whose gradient $\underset{\sim}{F}$ satisfies $(2.3)_1$. Thus, we must ask the following question: If $\underset{\sim}{C}$ is specified at every point of the body, can we find $\underset{\sim}{F}$ (or equivalently $\underset{\sim}{M}_K$) derivable from $\underset{\sim}{\chi}$ such that $(2.3)_1$ or (3.8) are satisfied? To this end, we proceed to express the derivative of $\underset{\sim}{M}_K$ in terms of the derivatives of $\underset{\sim}{C}$. First, we differentiate (3.8) with respect to X_M and obtain

*The definition of the transpose in $(3.6)_1$ is given by $(A2)_1$ of Appendix A.

$$C_{KL,M} = (\underset{\sim}{M}_K \cdot \underset{\sim}{M}_L)_{,M} = \underset{\sim}{M}_{K,M} \cdot \underset{\sim}{M}_L + \underset{\sim}{M}_K \cdot \underset{\sim}{M}_{L,M} \quad . \tag{3.9}$$

Next, by suitable combination of expressions of the type (3.9) and with the help of (3.7), the derivative of $\underset{\sim}{M}_K$ can be expressed in terms of partial derivatives of components C_{KL} alone, i.e.,

$$\underset{\sim}{M}_L \cdot \underset{\sim}{M}_{K,M} = F_{iL}F_{iK,M} = \tfrac{1}{2}(C_{LM,K} + C_{LK,M} - C_{KM,L}) \tag{3.10}$$

or equivalently in coordinate-free notation[††]

$$\underset{\sim}{F}^T \text{ Grad } \underset{\sim}{F} = [(\text{Grad } \underset{\sim}{C})^{T_1} + \text{Grad } \underset{\sim}{C} - (\text{Grad } \underset{\sim}{C})^{T_3}] \quad . \tag{3.11}$$

Thus, if there exists an $\underset{\sim}{F}$ which satisfies (3.6) and $(2.3)_1$, then $\underset{\sim}{F}$ must satisfy the differential equations (3.10) or (3.11). Clearly, the differential equations (3.10) or (3.11) are the necessary conditions for the determination of $\underset{\sim}{F}$ from the knowledge of $\underset{\sim}{C}$. The converse, proved in Appendix B, is[*]: if a field $\underset{\sim}{F}$ satisfies (3.10), then $(2.3)_1$ holds and furthermore a motion $\underset{\sim}{\chi}$ exists whose gradient is $\underset{\sim}{F}$.

Since the stretch tensor $\underset{\sim}{U}$ is uniquely determined by $\underset{\sim}{C}$, we may use (2.1) to obtain the following differential equations[§] for the rotation $\underset{\sim}{R}$ in terms of $\underset{\sim}{U}$:

$$\underset{\sim}{R}^T \text{ Grad } \underset{\sim}{R} = \underset{\sim}{\mathcal{A}} \quad , \tag{3.12}$$

where $\underset{\sim}{\mathcal{A}}$ is a third order tensor given by

[††] The notations T_1, T_2 and T_3 for the transpose of third order tensors is introduced in (A2) of Appendix A.

[*] This sufficiency argument is discussed in the first paragraph of Appendix B.

[§] A sketch of the derivation of the differential equations (3.12) or equivalently (3.14) is given in the second paragraph of Appendix B leading to (B4).

$$\underset{\sim}{\mathcal{A}} = -\underset{\sim}{\mathcal{A}}^{T_2} = \tfrac{1}{2}\{\underset{\sim}{U}^{-1}[\text{Grad }\underset{\sim}{U} - (\text{Grad }\underset{\sim}{U})^{T_3}]$$
$$+ ([\text{Grad }\underset{\sim}{U} - (\text{Grad }\underset{\sim}{U})^{T_1}]\underset{\sim}{U}^{-1})^{T_1}$$
$$+ (\underset{\sim}{U}^{-1}[\text{Grad }\underset{\sim}{U} - (\text{Grad }\underset{\sim}{U})^{T_3}]\underset{\sim}{U}^{-1})^{T_1}\underset{\sim}{U}\} \quad . \tag{3.13}$$

Thus, if $\underset{\sim}{R}$ exists such that (2.1) holds, then it is necessary that $\underset{\sim}{R}$ satisfy (3.12). Conversely, if a proper orthogonal tensor $\underset{\sim}{R}$ is a solution of (3.12), then by reversing the procedure used in obtaining (3.12) from (3.10), it can be shown that the tensor $\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$ satisfies (3.10) and the existence of a deformation function $\underset{\sim}{\chi}$ is ensured.

In the remainder of the paper, it is often more convenient to write (3.12) as

$$\underset{\sim}{R}^T\underset{\sim}{R}_{,K} = \underset{\sim}{A}_K \quad , \quad (K = 1,2,3) \quad , \tag{3.14}$$

where each $\underset{\sim}{A}_K$ $(K = 1,2,3)$ is a skew-symmetric second order tensor such that

$$\underset{\sim}{A}_K = -\underset{\sim}{A}_K^T = \underset{\sim}{\mathcal{A}}\,\underset{\sim}{e}_K$$
$$= \tfrac{1}{2}\{\underset{\sim}{U}^{-1}\underset{\sim}{U}_{,K} - \underset{\sim}{U}_{,K}\underset{\sim}{U}^{-1} + \underset{\sim}{U}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L)\underset{\sim}{U}^{-1}$$
$$- \underset{\sim}{U}^{-1}(\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{U}_{,L} + \underset{\sim}{U}^{-1}\underset{\sim}{U}_{,L}\underset{\sim}{U}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L)\underset{\sim}{U}^{-1}$$
$$- \underset{\sim}{U}^{-1}(\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{U}\,\underset{\sim}{U}_{,L}\underset{\sim}{U}^{-1}\} \quad . \tag{3.15}$$

It should be noted that the right-hand side of (3.13), or equivalently (3.15), involves only the stretch $\underset{\sim}{U}$ and its gradient.

Integrability conditions for the existence of a solution $\underset{\sim}{R}$ to (3.14) have been discussed by Shield (1973) based on a paper of Thomas (1934). A brief statement of these conditions in the context of the present paper is given following (B4) of Appendix B.

In the rest of this section, we state two theorems, the proofs of

which utilize the representation (3.14) or some of the preceding results. The first of these pertains to uniqueness of the rotation tensor to within a rigid body rotation and may be stated as

<u>Theorem 3.1</u>. Let a stretch field $\underset{\sim}{U}$ derivable from a motion $\underset{\sim}{\chi}$ exist such that the compatibility conditions (3.6) are satisfied and let $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$ be any two rotation tensors satisfying (2.2). Then, $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$ corresponding to $\underset{\sim}{U}$ are solutions of (3.14) if and only if they differ by a rigid body rotation.

<u>Proof</u>. We first prove the necessity. Suppose that $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$ are any two distinct proper orthogonal tensors, each of which satisfy (3.14) corresponding to the same stretch, i.e.,

$$\underset{\sim}{R}_1^T \underset{\sim}{R}_{1,K} = \underset{\sim}{A}_K \quad , \quad \underset{\sim}{R}_2^T \underset{\sim}{R}_{2,K} = \underset{\sim}{A}_K \quad . \tag{3.16}$$

Consider next the space derivative of the product $\underset{\sim}{R}_1 \underset{\sim}{R}_2^T$, namely

$$\begin{aligned} (\underset{\sim}{R}_1 \underset{\sim}{R}_2^T)_{,K} &= \underset{\sim}{R}_1 (\underset{\sim}{R}_1^T \underset{\sim}{R}_{1,K}) \underset{\sim}{R}_2^T + \underset{\sim}{R}_1 (\underset{\sim}{R}_{2,K}^T \underset{\sim}{R}_2) \underset{\sim}{R}_2^T \\ &= \underset{\sim}{R}_1 \underset{\sim}{A}_K \underset{\sim}{R}_2^T - \underset{\sim}{R}_1 \underset{\sim}{A}_K \underset{\sim}{R}_2^T \\ &= \underset{\sim}{0} \quad , \end{aligned} \tag{3.17}$$

where in obtaining $(3.17)_1$ the first term has been premultiplied by $\underset{\sim}{R}_1 \underset{\sim}{R}_1^T = \underset{\sim}{I}$ and the second term has been postmultiplied by $\underset{\sim}{R}_2 \underset{\sim}{R}_2^T = \underset{\sim}{I}$ and where use has been made of $(3.16)_{1,2}$ and $(3.15)_1$. It follows from (3.17) that the product $\underset{\sim}{R}_1 \underset{\sim}{R}_2^T$ must be a second order tensor which is independent of position but may be a function of time. Since each of the two tensors $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$ is proper orthogonal, the product $\underset{\sim}{R}_1 \underset{\sim}{R}_2^T$ is also proper orthogonal, say $\underset{\sim}{Q}_o(t)$. Thus, we may conclude that $\underset{\sim}{R}_1 \underset{\sim}{R}_2^T = \underset{\sim}{Q}_o$ or equivalently

$$\underset{\sim}{R}_1 = \underset{\sim}{Q}_o \underset{\sim}{R}_2 \quad . \tag{3.18}$$

Hence $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$ differ at most by a proper orthogonal tensor function of time corresponding to a superposed rigid body rotation. To show that the conclusion reached is also sufficient, we only need to replace $\underset{\sim}{R}$ in (3.14) by $\underset{\sim}{Q}_o\underset{\sim}{R}$ and the theorem is proved.

An immediate consequence of Theorem 3.1 is the following

Corollary 3.1. Corresponding to a given field of deformation $\underset{\sim}{C}$, the deformation gradient field $\underset{\sim}{F}$ can be determined to within a rigid body rotation.

The proof follows at once by recalling (2.1) and noting that corresponding to a given $\underset{\sim}{C}$ the tensor $\underset{\sim}{U}$ is uniquely determined by $(2.3)_2$. A proof of this result was apparently first given by Shield (1973, p. 484) who employs a different procedure than that of our Theorem 3.1 and does not make use of (3.14).

In the statement of Theorem 3.1, the existence of a motion $\underset{\sim}{\chi}$ whose stretch field is $\underset{\sim}{U}$ was assumed. Suppose instead that a field $\underset{\sim}{U}$ is prescribed as a function of $\underset{\sim}{X}$. It is then natural to ask under what conditions such a stretch is the gradient of some deformation function $\underset{\sim}{\chi}$. In other words, what restrictions must be placed on $\underset{\sim}{U}$ in order to guarantee that the deformation gradient is of the form

$$\underset{\sim}{F} = \underset{\sim}{U} \quad \text{or} \quad F_{iK} = \delta_{iL}U_{LK} \quad . \tag{3.19}$$

This leads us to state the following

Theorem 3.2. The restrictions

$$\text{Grad } \underset{\sim}{U} = (\text{Grad } \underset{\sim}{U})^{T_1} \quad \text{or} \quad U_{KL,M} = U_{KM,L} \tag{3.20}$$

on $\underset{\sim}{U}$ are both necessary and sufficient to ensure the existence of a motion whose gradient is symmetric and hence corresponds to pure stretch.

Proof. Suppose a motion exists whose gradient satisfies (3.19). Then, the compatibility conditions (3.6) imply the restrictions (3.20) as necessary conditions for the existence of $\underset{\sim}{\chi}$. To show sufficiency, assume the restrictions (3.20) on $\underset{\sim}{U}$. With the use of (3.20), it can then be shown that the right-hand side of (3.15) vanishes identically* and hence $\underset{\sim}{A}_K = \underset{\sim}{0}$. Further, by (3.14) we have $\underset{\sim}{R}^T \underset{\sim}{R}_{,K} = \underset{\sim}{0}$. This, in turn, implies that $\underset{\sim}{R}_{,K} = \underset{\sim}{0}$ since $\underset{\sim}{R}$ is nonsingular. Hence, $\underset{\sim}{R}$ must be a function of time only corresponding to a superposed rigid body rotation, say $\underset{\sim}{Q}_o$. Thus, a motion $\underset{\sim}{Q}_o^T \underset{\sim}{\chi}$ whose gradient is $\underset{\sim}{Q}_o^T \underset{\sim}{F}$ is sufficient to satisfy (3.19) and the theorem is proved.

4. Kinematics of deformation with moderate local rotation

We are concerned here with a kinematical development in which the Lagrangian strain is infinitesimal but is not necessarily accompanied by infinitesimal rotation. To make these notions precise, in a manner similar to that of Casey and Naghdi (1981), we first define a measure of smallness of strain by the nonnegative real function[§]

$$\varepsilon_0 = \varepsilon_0(t) = \sup_{\underset{\sim}{X} \in \mathcal{R}_0} \|\underset{\sim}{E}(\underset{\sim}{X},t)\| \quad , \tag{4.1}$$

where sup stands for the supremum (or least upper bound) of a nonempty bounded set of real numbers. If $h_0(\underset{\sim}{E})$ is any scalar-, vector-, or tensor-valued function of $\underset{\sim}{E}$ defined in the neighborhood of $\underset{\sim}{E} = \underset{\sim}{0}$ and satisfying the condition that there exists a nonnegative real constant C such that $\|h_0(\underset{\sim}{E})\| < C\varepsilon_0^n$ as $\varepsilon_0 \to 0$, then we write $h_0 = O(\varepsilon_0^n)$ as $\varepsilon_0 \to 0$.

The statement $\underset{\sim}{E} = \underset{\sim}{O}(\varepsilon_0)$ as $\varepsilon_0 \to 0$ does not imply any restriction on the space or time derivatives of $\underset{\sim}{E}$. In particular, it is possible for $\text{Grad}\,\underset{\sim}{E} = \underset{\sim}{E}_{,K} \otimes \underset{\sim}{e}_K$ or equivalently $\underset{\sim}{E}_{,K}$ to be finite while $\underset{\sim}{E}$ itself remains small. To deal with such circumstances, we introduce a second measure of smallness by a nonnegative real function[*]

$$\varepsilon_1 = \varepsilon_1(t) = \max_K \sup_{\underset{\sim}{X} \in \mathcal{R}_0} \|\underset{\sim}{E}_{,K}(\underset{\sim}{X},t)\| \quad , \quad (K = 1,2,3) \quad . \tag{4.2}$$

If $h_1(\underset{\sim}{E}_{,K})$ is any scalar-, vector-, or tensor-valued function of Grad $\underset{\sim}{E}$

[§] In the paper of Casey and Naghdi (1981), a quantity corresponding to ε_0 was defined in terms of the displacement gradient rather than the strain.

[*] In writing (4.2), for simplicity we have used the norm of the second order tensors $\underset{\sim}{E}_{,K}$ defined in section 1. It is, of course, possible to define ε_1 in terms of Grad $\underset{\sim}{E}$, but this requires also a definition for the norm of a third order tensor which is not introduced in section 1 or Appendix A.

defined in the neighborhood of $\underset{\sim}{E}_{,K} = (\underset{\sim}{E}_{,1}, \underset{\sim}{E}_{,2}, \underset{\sim}{E}_{,3}) = \underset{\sim}{0}$ satisfying the condition that there exists a positive real constant D such that $||h_1(\underset{\sim}{E}_{,K})|| < D\varepsilon_1^n$ as $\varepsilon_1 \to 0$, then we write $h_1 = O(\varepsilon_1^n)$ as $\varepsilon_1 \to 0$. Further, if there exists a nonnegative constant $\overline{C}$ such that

$$\varepsilon_1^k < \overline{C}\varepsilon_o^\ell \quad \text{with} \quad k,\ell \text{ integers} \quad , \tag{4.3}$$

then it can be shown that*

$$||h_1(\underset{\sim}{E}_{,K})|| < (D\overline{C}^{n/k})\varepsilon_o^{n\ell/k} \quad . \tag{4.4}$$

Provided that (4.3) holds, the last result implies that any function $h_1(\underset{\sim}{E}_{,K})$ of $O(\varepsilon_1^n)$ as $\varepsilon_1 \to 0$ is also of $O(\varepsilon_o^{n\ell/k})$ as $\varepsilon_o \to 0$.

We now introduce two additional measures of smallness, one associated with the rotation tensor and another with its gradients. Thus we define the measures of smallness ε_2 and ε_3, respectively, by the nonnegative real functions

$$\varepsilon_2 = \varepsilon_2(t) = \sup_{\underset{\sim}{X} \in \mathcal{R}_o} ||\underset{\sim}{R} - \underset{\sim}{I}|| \tag{4.5}$$

and

$$\varepsilon_3 = \varepsilon_3(t) = \max_K\{\sup_{\underset{\sim}{X} \in \mathcal{R}_o} ||\underset{\sim}{R}_{,K}||\} \quad , \quad (K = 1,2,3) \quad . \tag{4.6}$$

If $h_2(\underset{\sim}{R}-\underset{\sim}{I})$ is any scalar-, vector-, or tensor-valued function of $\underset{\sim}{R}$ defined in the neighborhood of $\underset{\sim}{R} = \underset{\sim}{I}$ satisfying the condition that there exists a positive real constant D_2 such that $||h_2(\underset{\sim}{R}-\underset{\sim}{I})|| < D_2\varepsilon_2^n$ as $\varepsilon_2 \to 0$, then we write $h_2 = O(\varepsilon_2^n)$ as $\varepsilon_2 \to 0$. Similarly, if $h_3(\underset{\sim}{R}_{,K})$ is any scalar-, vector-, or tensor-valued function of $\underset{\sim}{R}_{,K}$ $(K = 1,2,3)$ defined in the neighborhood

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*The details are discussed following (B8) in Appendix B.

of $\underset{\sim}{R}_{,K} = 0$ satisfying the condition that there exists a positive real constant D_3 such that $\|h_3(\underset{\sim}{R}_{,K})\| < D_3\varepsilon_3^n$ as $\varepsilon_3 \to 0$, then we write $h_3 = 0(\varepsilon_3^n)$ as $\varepsilon_3 \to 0$. Further, the four measures ε_m, $(m = o,1,2,3)$ can be related if there exist nonnegative constants $\overline{C}_{mn}$, $m,n = o,1,2,3$, such that

$$\varepsilon_m^k < \overline{C}_{mn}\varepsilon_n^\ell \quad \text{with} \quad k,\ell \text{ integers} \quad , \tag{4.7}$$

and then similar to (4.4) it can be shown that any function of $0(\varepsilon_m^n)$ as $\varepsilon_m \to 0$ is also of $0(\varepsilon_n^{n\ell/k})$ as $\varepsilon_n \to 0$.

Since $\underset{\sim}{R}$ is a proper orthogonal tensor, it follows from the definition of the form of a second order tensor that

$$\|\underset{\sim}{R}-\underset{\sim}{I}\|^2 = \mathrm{tr}\{(\underset{\sim}{R}-\underset{\sim}{I})^T(\underset{\sim}{R}-\underset{\sim}{I})\} = 2(3 - \mathrm{tr}\,\underset{\sim}{R}) \quad . \tag{4.8}$$

Recalling the expression for tr $\underset{\sim}{R}$ in (2.15), which implies that tr $\underset{\sim}{R}$ satisfies the inequality $-1 \leq \mathrm{tr}\,\underset{\sim}{R} \leq 3$, it follows from (4.8) that $0 \leq \|\underset{\sim}{R}-\underset{\sim}{I}\|^2 \leq 8$ and consequently the measure of smallness ε_2 in (4.5) is bounded from above by $2\sqrt{2}$, i.e., $\varepsilon_2 \leq 2\sqrt{2}$.

A solution of (3.14) involves both $\underset{\sim}{U}$ and its derivatives $\underset{\sim}{U}_{,K}$. In order to accommodate the strains and strain gradients of different orders of magnitude, we first express $\underset{\sim}{U}_{,K}$ in terms of $\underset{\sim}{E}$ and $\underset{\sim}{E}_{,K}$. Remebering that by $(2.3)_2$ and (2.4) the stretch $\underset{\sim}{U}$ may be regarded as a function of $\underset{\sim}{E}$, from the Taylor expansion of $\underset{\sim}{U}(\underset{\sim}{E})$ about $\underset{\sim}{E} = \underset{\sim}{0}$ follows**

$$\underset{\sim}{U} = \underset{\sim}{I} + \underset{\sim}{E} - \frac{1}{2!}\underset{\sim}{E}^2 + \frac{1}{3!}(3\underset{\sim}{E}^3) + \cdots \tag{4.9}$$

**A statement of Taylor's formula of the form (4.9) is given by Theorem 8.14.3 of Dieudonné (1969, p. 190). Since all the spaces considered here are real Euclidean spaces and since all finite dimensional Euclidean spaces are Banach spaces, the theorem in (Dieudonné 1969) is directly applicable.

and by differentiation we have

$$\underset{\sim}{U}_{,K} = \underset{\sim}{E}_{,K} - \tfrac{1}{2}(\underset{\sim}{E}_{,K}\underset{\sim}{E} + \underset{\sim}{E}\underset{\sim}{E}_{,K}) + \tfrac{1}{2}[\underset{\sim}{E}_{,K}\underset{\sim}{E}^2 + \underset{\sim}{E}\underset{\sim}{E}_{,K}\underset{\sim}{E} + \underset{\sim}{E}^2\underset{\sim}{E}_{,K}] + \cdots \quad , \tag{4.10}$$

where in writing (4.9) we have also used the fact that $\underset{\sim}{U} = \underset{\sim}{I}$ at $\underset{\sim}{E} = \underset{\sim}{0}$. Before proceeding further, in the context of the classical infinitesimal kinematics, we recall the approximate formulae for $\underset{\sim}{U}, \underset{\sim}{C}$ and their inverses, all estimated in terms of the infinitesimal Lagrangian strain, i.e.,

$$\begin{aligned} &\underset{\sim}{E} = \underset{\sim}{0}(\varepsilon_o) \ , \quad \underset{\sim}{U} = \underset{\sim}{I} + \underset{\sim}{E} = \underset{\sim}{I} + \underset{\sim}{0}(\varepsilon_o) \ , \quad \underset{\sim}{U}^{-1} = \underset{\sim}{I} - \underset{\sim}{E} = \underset{\sim}{I} - \underset{\sim}{0}(\varepsilon_o) \ , \\ &\underset{\sim}{C} = \underset{\sim}{I} + 2\underset{\sim}{E} = \underset{\sim}{I} + \underset{\sim}{0}(\varepsilon_o) \ , \quad \underset{\sim}{C}^{-1} = \underset{\sim}{I} - 2\underset{\sim}{E} = \underset{\sim}{I} - \underset{\sim}{0}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \ . \end{aligned} \tag{4.11}$$

Now suppose that $\underset{\sim}{E} = \underset{\sim}{0}(\varepsilon_o)$ as $\varepsilon_o \to 0$ in (4.10) but as yet impose no restriction on $\underset{\sim}{E}_{,K}$. It follows that each term of the last bracket in (4.10) is of $\underset{\sim}{0}(\varepsilon_o^2)$ and, to the order ε_o^2, the expression for $\underset{\sim}{U}_{,K}$ can be approximated by

$$\underset{\sim}{U}_{,K} = \underset{\sim}{E}_{,K} - \tfrac{1}{2}(\underset{\sim}{E}_{,K}\underset{\sim}{E} + \underset{\sim}{E}\underset{\sim}{E}_{,K}) = \underset{\sim}{E}_{,K} + 0(\varepsilon_o) \quad , \quad \text{as} \quad \varepsilon_o \to 0 \ . \tag{4.12}$$

We observe that with $\underset{\sim}{U}$ and $\underset{\sim}{U}_{,K}$ given by $(4.11)_2$ and (4.12), it can be verified that the expression $(\underset{\sim}{U}\underset{\sim}{U})_{,K} = 2\underset{\sim}{E}_{,K}$ and thus $(4.11)_2$ and (4.12) satisfy $(2.8)_2$ to the order ε_o. Introducing the approximations (4.11) and (4.12) on the right-hand side of (3.15), we obtain

$$\begin{aligned} \underset{\sim}{A}_K = -\underset{\sim}{A}_K^T = {} & \underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L) - (\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_{,L} \\ & + \tfrac{1}{2}\{\underset{\sim}{E}_{,K}\underset{\sim}{E} - \underset{\sim}{E}\underset{\sim}{E}_{,K} - 2[\underset{\sim}{E}\underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L) - (\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_{,L}\underset{\sim}{E}] \\ & + 2[\underset{\sim}{E}(\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_{,L} - \underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L)\underset{\sim}{E}]\} \quad , \end{aligned} \tag{4.13}$$

where terms of $\underset{\sim}{0}(\varepsilon_o^2)$ have been neglected.

4.1 Estimate for the angle of rotation

With the help of (1.3), we can estimate the norm of $\underset{\sim}{A}_K$ from the expression (4.13):

$$\|\underset{\sim}{A}_K\| < \|\underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L)\| + \|(\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_L\| + \tfrac{1}{2}\{\|\underset{\sim}{E}_{,K}\underset{\sim}{E}\|$$

$$+ \|\underset{\sim}{E}\underset{\sim}{E}_{,K}\| + 2\|\underset{\sim}{E}\underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L)\| + 2\|(\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_{,L}\underset{\sim}{E}\|$$

$$+ 2\|\underset{\sim}{E}(\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_{,L}\| + 2\|\underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L)\underset{\sim}{E}\|\}$$

$$< 6\varepsilon_1 + 13\varepsilon_o\varepsilon_1 = \varepsilon_1(6 + 13\varepsilon_o) \quad . \tag{4.14}$$

Hence, given ε_o, it follows from (4.14) that

$$\underset{\sim}{A}_K = \underset{\sim}{0}(\varepsilon_1) \quad \text{as} \quad \varepsilon_1 \to 0 \quad , \quad (K = 1,2,3) \quad . \tag{4.15}$$

From (3.14), which can be written as $\underset{\sim}{R}_{,K} = \underset{\sim}{R}\underset{\sim}{A}_K$, the norm of $\underset{\sim}{R}_{,K}$ is given by

$$\|\underset{\sim}{R}_{,K}\|^2 = \mathrm{tr}\{(\underset{\sim}{R}_{,K})^T\underset{\sim}{R}_{,K}\} = \|\underset{\sim}{A}_K\|^2 \quad . \tag{4.16}$$

By (4.16) and (4.14) we also have $\|\underset{\sim}{R}_{,K}\| < \varepsilon_1(6 + 13\varepsilon_o)$ and it follows that the order of magnitude of $\underset{\sim}{R}_{,K}$ is given by

$$\underset{\sim}{R}_{,K} = \underset{\sim}{0}(\varepsilon_1) \quad \text{as} \quad \varepsilon_1 \to 0 \quad , \quad (K = 1,2,3) \quad . \tag{4.17}$$

In terms of (4.7) with $m = 3$ and $n = 1$, the estimate $\|\underset{\sim}{R}_{,K}\|$ noted above implies (4.7) with $k = \ell = 1$, and we conclude that quantities of $0(\varepsilon_3)$ are also of $0(\varepsilon_1)$ as $\varepsilon_1 \to 0$. It should be noted that although (3.14), which involves $\underset{\sim}{R}$, is used to estimate the order of magnitude of $\underset{\sim}{R}_{,K}$, the result (4.17) is independent of the measure of smallness ε_2.

In order to assess the effect of $\underset{\sim}{E}_{,K}$ on the order of magnitude of the angle of rotation, we now use the representation (2.13) to calculate the left-hand side of (3.14):

$$\begin{aligned}\underset{\sim}{R}^T\underset{\sim}{R}_{,K} = {} & (1-\cos\theta)(\underset{\sim}{\mu}\otimes\underset{\sim}{\mu}_{,K}-\underset{\sim}{\mu}_{,K}\otimes\underset{\sim}{\mu}) \\ & + \sin\theta(1-\cos\theta)\{(\underset{\sim}{\mu}_{,K}\times\underset{\sim}{\mu})\otimes\underset{\sim}{\mu}-\underset{\sim}{\mu}\otimes(\underset{\sim}{\mu}_{,K}\times\underset{\sim}{\mu})\} \\ & - \theta_{,K}\underset{\sim}{e}_L\otimes(\underset{\sim}{\mu}\times\underset{\sim}{e}_L) - \sin\theta\cos\theta\;\underset{\sim}{e}_L\otimes(\underset{\sim}{\mu}_{,K}\times\underset{\sim}{e}_L) \quad .\end{aligned} \tag{4.18}$$

The above expression involves both the angle θ and its gradients. In order to estimate the order of magnitude of the latter quantities, we proceed to isolate the term containing $\theta_{,K}$ in (4.18). To this end, we first observe that if the directions $\underset{\sim}{\mu}_{,K}$ vanish, every term in (4.18) is zero except the one containing $\theta_{,K}$, namely $-\theta_{,K}\underset{\sim}{e}_L\otimes(\underset{\sim}{\mu}\times\underset{\sim}{e}_L)$. Next, let $\underset{\sim}{\alpha}_K$ $(K=1,2,3)$ be a set of unit vectors along the directions of $\underset{\sim}{\mu}_{,K}\neq 0$ and choose a second set of unit vectors $\underset{\sim}{\gamma}_K$ $(K=1,2,3)$ such that for each K the set of unit vectors $(\underset{\sim}{\mu},\underset{\sim}{\gamma}_K,\underset{\sim}{\alpha}_K)$ form a right-handed orthogonal triad*. Then, by considering the scalar $\underset{\sim}{\alpha}_K\cdot\underset{\sim}{R}^T\underset{\sim}{R}_{,K}\underset{\sim}{\gamma}_K$ (no sum on K), it can be verified that**

*Our preference for the order of the unit vectors in the right-handed triad $(\underset{\sim}{\mu},\underset{\sim}{\gamma}_K,\underset{\sim}{\alpha}_K)$ is simply because this leads to a positive sign on the right-hand side of (4.19). Alternatively, a negative sign would result on the right-hand side of (4.19) if we had chosen the set $(\underset{\sim}{\mu},\underset{\sim}{\alpha}_K,\underset{\sim}{\gamma}_K)$ as a right-handed orthogonal triad.

**Details are given following (B10) in Appendix B.

$$\underset{\sim}{\alpha}_K \cdot \underset{\sim}{R}^T \underset{\sim}{R}_{,K} \underset{\sim}{\gamma}_K = -\theta_{,K}(\underset{\sim}{\mu} \times \underset{\sim}{\alpha}_K \cdot \underset{\sim}{\gamma}_K) = +\theta_{,K} \quad \text{(no sum on K)} \quad . \tag{4.19}$$

From comparison of (3.14), (4.15) and (4.19) we may now conclude that throughout the body $\mathscr{B}$ the gradients of the angle of rotation have the order of magnitude given by

$$\theta_{,K} = O(\varepsilon_1) \quad \text{as} \quad \varepsilon_1 \to 0 \quad (K = 1,2,3) \quad . \tag{4.20}$$

To continue the discussion, suppose that $\underset{\sim}{R} = \underset{\sim}{I}$ at some material point ${}_o\underset{\sim}{X} \in \mathscr{B}$, or that equivalently $\theta = 0$ at ${}_o\underset{\sim}{X}$. Then, at any other material point $\underset{\sim}{X}$, the angle of rotation can be calculated from

$$\theta = \int_{{}_o\underset{\sim}{X}}^{\underset{\sim}{X}} (\partial\theta/\partial X_K)\lambda_K \, dS \quad , \tag{4.21}$$

where dS is the arc length of an arbitrary curve C in the reference configuration of $\mathscr{B}$, $\underset{\sim}{\lambda} = \lambda_K \underset{\sim}{e}_K$ is the unit tangent vector to C and the integration is performed along the curve joining* ${}_o\underset{\sim}{X}$ and $\underset{\sim}{X}$. By means of the usual inequalities for integrals, from (4.21) we obtain an estimate for the magnitude of θ in the form

$$|\theta| \leq \int_{{}_o\underset{\sim}{X}}^{\underset{\sim}{X}} |\partial\theta/\partial X_K| \; |\lambda_K| dS = |\overline{\theta_{,K}}| \; |\overline{\lambda_K}| L \quad , \tag{4.22}$$

where $(\overline{\theta_{,K}})$ and $\overline{\lambda_K}$ are the values of $\theta_{,K}$ and λ_K at some point on the curve C, respectively, and where L is the length of the curve C joining ${}_o\underset{\sim}{X}$ and $\underset{\sim}{X}$. Since $\overline{\lambda_K}$ are components of a unit vector and since the estimate (4.20) holds throughout the body, from (4.22) we conclude that the angle θ at every point of $\mathscr{B}$ has the order of magnitude

$$\theta = O(\varepsilon_1) \quad \text{as} \quad \varepsilon_1 \to 0 \quad , \tag{4.23}$$

provided that $\underset{\sim}{R} = \underset{\sim}{I}$ at some point ${}_o\underset{\sim}{X} \in \mathscr{B}$. Recalling the power series

*The existence of such a curve in $\mathscr{B}$ is ensured by the connectivity assumption of the region $\mathscr{R}_o$ in section 2.

expansions

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots \quad , \quad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \quad , \tag{4.24}$$

it follows from the estimate (4.23) that

$$\sin\theta = 0(\varepsilon_1) \quad , \quad \cos\theta = 1 + 0(\varepsilon_1^2) \quad \text{as} \quad \varepsilon_1 \to 0 \quad . \tag{4.25}$$

Using (4.25) in the representation (2.13), we conclude that provided $\underset{\sim}{R} = \underset{\sim}{I}$ at ${}_{o}\underset{\sim}{X}$, the rotation tensor $\underset{\sim}{R}$ satisfies

$$\underset{\sim}{R} = \underset{\sim}{I} + \underset{\sim}{0}(\varepsilon_1) \quad \text{as} \quad \varepsilon_1 \to 0 \quad . \tag{4.26}$$

Since by (4.5) the rotation was defined as $\underset{\sim}{R} = \underset{\sim}{I} + \underset{\sim}{0}(\varepsilon_2)$, it follows from (4.26) that provided $\underset{\sim}{R}$ is a unit tensor at some point ${}_{o}\underset{\sim}{X}$ of the body, quantities of $0(\varepsilon_2)$ are comparable to $0(\varepsilon_1)$ as $\varepsilon_1 \to 0$.

Thus far in the development of this section, the order of magnitude of $\underset{\sim}{E}_{,K}$ (and hence also of $\underset{\sim}{R}$ or θ) has been regarded as independent of that of the strain $\underset{\sim}{E}$. In order to relate the estimate of the rotation tensor $\underset{\sim}{R}$ in (4.26) or the angle of rotation θ in (4.23) to the order of magnitude ε_o, an additional assumption must be made concerning the relationship between the orders of magnitude of $\underset{\sim}{E}$ and $\underset{\sim}{E}_{,K}$ or equivalently between ε_o and ε_1. But, prior to such an undertaking, we need to dispose of some geometrical preliminaries and definitions.

Consider now any unit vector $\underset{\sim}{v}$ and rotate this by the rotation tensor $\underset{\sim}{R}$ through an angle α resulting in the vector $\underset{\sim}{R}\,\underset{\sim}{v}$. Since the proper orthogonal tensor $\underset{\sim}{R}$ is length preserving, the magnitude of $\underset{\sim}{v}$ remains unchanged upon rotation and we have

$$\cos\alpha = \underset{\sim}{v} \cdot \underset{\sim}{R}\,\underset{\sim}{v} \quad , \tag{4.27}$$

which represents the projection of the rotated vector $\underset{\sim}{R}\,\underset{\sim}{v}$ along $\underset{\sim}{v}$ (see the

sketch in Fig. 1). With the use of (2.13), it can be shown from (4.27) that $\cos\alpha$ is bounded from below by $\cos\theta$ in (2.14), i.e.*,

$$\cos\alpha \geq \cos\theta \quad . \tag{4.28}$$

It may be noted here that the angle $\alpha = 0$ if $\underset{\sim}{v}$ is parallel to the unit vector $\underset{\sim}{\mu}$ in (2.14), while $\alpha = \theta$ if $\underset{\sim}{v}$ is perpendicular to $\underset{\sim}{\mu}$.

To continue the discussion, let $\underset{\sim}{\beta}$ stand for a vector defined by (see also the sketch in Fig. 1)

$$\underset{\sim}{\beta} = \underset{\sim}{R}\,\underset{\sim}{v} - \underset{\sim}{v} \quad \text{or} \quad \underset{\sim}{R}\,\underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{\beta} \quad . \tag{4.29}$$

Keeping in mind that both $\underset{\sim}{v}$ and $\underset{\sim}{R}\,\underset{\sim}{v}$ are unit vectors, from the inner product of (4.29) with itself we arrive at

$$\tfrac{1}{2}\underset{\sim}{\beta}\cdot\underset{\sim}{\beta} = \tfrac{1}{2}|\underset{\sim}{\beta}|^2 = 1 - \underset{\sim}{v}\cdot\underset{\sim}{R}\,\underset{\sim}{v} = 1 - \cos\alpha \quad , \tag{4.30}$$

which shows that $\tfrac{1}{2}|\underset{\sim}{\beta}|^2$ is the difference between unity and the projection of $\underset{\sim}{R}\,\underset{\sim}{v}$ along $\underset{\sim}{v}$. Now, let ϕ denote the angle that the vector $\underset{\sim}{\beta}$ makes with $\underset{\sim}{v}$ as indicated in Fig. 1. Then, by $(4.29)_1$, we have

$$\underset{\sim}{\beta}\cdot\underset{\sim}{v} = |\underset{\sim}{\beta}|\cos\phi = -(1 - \underset{\sim}{v}\cdot\underset{\sim}{R}\,\underset{\sim}{v}) \quad . \tag{4.31}$$

Since the right-hand side of (4.31) is $-\tfrac{1}{2}|\underset{\sim}{\beta}|^2$ by $(4.30)_2$, it follows that $\cos\phi$ in (4.31) is not an independent quantity and is, in fact, given by $\cos\phi = -\tfrac{1}{2}|\underset{\sim}{\beta}|$.

4.2 The special case of infinitesimal kinematics

Before proceeding further, it is instructive to consider the case of classical infinitesimal kinematics in which both the strain and rotation are small. Thus, we introduce the following

*Details of the argument are given following (B11) in Appendix B.

Definition 4.1. Given $\underset{\sim}{E} = \underset{\sim}{0}(\varepsilon_o)$, a proper orthogonal tensor $\underset{\sim}{R}$ is said to be an infinitesimal rotation with respect to ε_o if for any unit vector $\underset{\sim}{v}$, the vector $\underset{\sim}{\beta}$ defined in (4.29) satisfies

$$\underset{\sim}{\beta} = \underset{\sim}{0}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \ . \tag{4.32}$$

It follows at once from (4.32) and $(4.30)_2$ that $\underset{\sim}{R}$ is an infinitesimal rotation in the sense of (4.32) if and only if

$$1 - \underset{\sim}{v} \cdot \underset{\sim}{R}\,\underset{\sim}{v} = 0(\varepsilon_o^2) \quad \text{as} \quad \varepsilon_o \to 0 \ , \tag{4.33}$$

i.e., the projection of $\underset{\sim}{R}\,\underset{\sim}{v}$ along $\underset{\sim}{v}$ differs from unity by $0(\varepsilon_o^2)$ as $\varepsilon_o \to 0$ if and only if $\underset{\sim}{R}$ is an infinitesimal rotation. Clearly, (4.33) can be used to state an alternative definition of infinitesimal rotation which is equivalent to (4.32).

Observing from $(4.30)_3$ and (4.33) that $\cos\alpha = 1 + 0(\varepsilon_o^2)$ as $\varepsilon_o \to 0$, the inequality (4.28) together with the fact that $\alpha = \theta$ for some $\underset{\sim}{v}$ imply that

$$\begin{gathered} \cos\theta = 1 + 0(\varepsilon_o^2) \ , \quad \sin^2\theta = 0(\varepsilon_o^2) \ , \quad \sin\theta = 0(\varepsilon_o) \ , \\ \theta = 0(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \ . \end{gathered} \tag{4.34}$$

Introduction of the approximation (4.34) into $(2.14)_{1,2}$ yields

$$\underset{\sim}{\Phi} = \underset{\sim}{\Phi}^T = \underset{\sim}{0}(\varepsilon_o^2) \ , \quad \underset{\sim}{\Psi} = \underset{\sim}{\Omega} = -\underset{\sim}{\Omega}^T = \underset{\sim}{0}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \ . \tag{4.35}$$

If $\underset{\sim}{R}$ is an infinitesimal rotation with respect to ε_o everywhere in the body, then in terms of the condition (4.7) we have $m = 2$, $n = 0$, $k = \ell = 1$, and quantities of $0(\varepsilon_2)$ are also of $0(\varepsilon_o)$ as $\varepsilon_o \to 0$.

It should be emphasized that implicit in the Definition 4.1 is the assumption that the order of magnitude of $\underset{\sim}{R}$ can be estimated in terms of the order of magnitude of $\underset{\sim}{E}$. To avoid undue complications, we have postponed making explicit any relationship between ε_1 and ε_0 but the conditions under which $\underset{\sim}{R}$ can be estimated in terms of $\underset{\sim}{E}$ will be examined later. In this connection, it may be recalled that in the usual kinematics of the infinitesimal theory the restriction of smallness is imposed on the relative displacement gradient $\underset{\sim}{H}$ defined by (2.6); and then, it follows at once that both the strain and the rotation are infinitesimal. The approach in the Definition 4.1 differs from the usual in that so far no restriction has been placed on the relative displacement gradient $\underset{\sim}{H}$, but infinitesimal rotation is defined with respect to ε_0.

In the context of infinitesimal kinematics, we now state the following

Theorem 4.1. The relative deformation gradient $\underset{\sim}{H}$ is of $\underset{\sim}{O}(\varepsilon_0)$ as $\varepsilon_0 \to 0$ if and only if $\underset{\sim}{E} = \underset{\sim}{U} - \underset{\sim}{I} = \underset{\sim}{O}(\varepsilon_0)$ as $\varepsilon_0 \to 0$ and $\underset{\sim}{R}$ is infinitesimal rotation with respect to ε_0 in the sense of Definition 4.1.

Proof. Let the tensor $\underset{\sim}{U}$ be specified by $(4.11)_2$ and suppose that $\underset{\sim}{\Phi}, \underset{\sim}{\Psi}$ are given by (4.35) so that, after the neglect of terms of $\underset{\sim}{O}(\varepsilon_0^2)$, the rotation $\underset{\sim}{R} = \underset{\sim}{I} + \underset{\sim}{O}(\varepsilon_0)$ as $\varepsilon_0 \to 0$. Then, by (2.1) the deformation gradient $\underset{\sim}{F} = \underset{\sim}{I} + \underset{\sim}{O}(\varepsilon_0)$ as $\varepsilon_0 \to 0$ and $\underset{\sim}{H} = \underset{\sim}{O}(\varepsilon_0)$ as $\varepsilon_0 \to 0$.

Conversely, if $\underset{\sim}{H}$ defined by (2.6) is of $\underset{\sim}{O}(\varepsilon_0)$ as $\varepsilon_0 \to 0$, then from $(2.3)_2$ and $(2.6)_1$ we have

$$\underset{\sim}{U}^2 = \underset{\sim}{I} + \underset{\sim}{H} + \underset{\sim}{H}^T + \underset{\sim}{0}(\varepsilon_o^2) \quad \text{as} \quad \varepsilon_o \to 0 \; . \tag{4.36}$$

If terms of $\underset{\sim}{0}(\varepsilon_o^2)$ are neglected as $\varepsilon_o \to 0$ in (4.36), we have

$$\underset{\sim}{U} = \underset{\sim}{I} + \tfrac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T) = \underset{\sim}{I} + \underset{\sim}{0}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \; . \tag{4.37}$$

Recall now that $\underset{\sim}{R} = \underset{\sim}{F}\underset{\sim}{U}^{-1}$ and, with the help of (4.37), obtain

$$\begin{aligned} \underset{\sim}{R} &= \underset{\sim}{F}[\underset{\sim}{I} - \tfrac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T)] = (\underset{\sim}{I} + \underset{\sim}{H})(\underset{\sim}{I} - \tfrac{1}{2}\underset{\sim}{H} - \tfrac{1}{2}\underset{\sim}{H}^T) \\ &= \underset{\sim}{I} + \tfrac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T) = \underset{\sim}{I} + \underset{\sim}{0}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \; , \end{aligned} \tag{4.38}$$

where terms of $\underset{\sim}{0}(\varepsilon_o^2)$ have been neglected as $\varepsilon_o \to 0$. This completes the proof.

It may be worth recalling that the strain gradients $\underset{\sim}{E}_{,K} = \underset{\sim}{0}(\varepsilon_1)$ as $\varepsilon_1 \to 0$, in view of (4.2); and that subsequently the use of this measure of smallness, along with (4.18) enabled us to establish the estimate (4.23) for the angle of rotation θ provided that $\underset{\sim}{R} = \underset{\sim}{I}$ at some point ${}_o\underset{\sim}{X} \in \mathcal{B}$. The estimate (4.23) can be brought into correspondence with $(4.34)_4$, which results from Definition 4.1, if and only if functions of $0(\varepsilon_1)$ are assumed to be comparable to $0(\varepsilon_o)$. In terms of the conditions stated immediately after (4.4), we may choose $k = 1$, $\ell = 1$ so that any function of $0(\varepsilon_1) = 0(\varepsilon_o)$ as $\varepsilon_o \to 0$. With this additional assumption, we may now write

$$\underset{\sim}{E}_{,K} = \underset{\sim}{0}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \tag{4.39}$$

and this is consistent with

$$\theta = 0(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \; , \tag{4.40}$$

provided $\underset{\sim}{R} = \underset{\sim}{I}$ a· some point ${}_o\underset{\sim}{X} \in \mathcal{B}$.

We are now in a position to state the following

<u>Theorem 4.2</u>. Given $\underset{\sim}{E} = \underset{\sim}{0}(\varepsilon_o)$ and $\underset{\sim}{E}_{,K} = \underset{\sim}{0}(\varepsilon_o)$ as $\varepsilon_o \to 0$, the tensor $\underset{\sim}{R}$ associated with the deformation function $\underset{\sim}{\chi}$ is an infinitesimal rotation with respect to ε_o to within a rigid body rotation.

<u>Proof</u>. Provided that $\underset{\sim}{R} = \underset{\sim}{I}$ at some point ${}_o\underset{\sim}{X} \in \mathscr{B}$, the conclusion (4.40) implies that $\underset{\sim}{R}$ is an infinitesimal rotation with respect to ε_o. Thus with the use of (2.13) and (4.35), to the order of approximation considered, the infinitesimal rotation is given by $(\underset{\sim}{I} + \underset{\sim}{\Omega})$ as $\varepsilon_o \to 0$. Moreover, by Theorem 3.1 any other solution of (3.14) differs from $(\underset{\sim}{I} + \underset{\sim}{\Omega})$ by a proper orthogonal tensor function of time (say $\underset{\sim}{R}_o$) corresponding to a rigid body rotation and we have

$$\underset{\sim}{R} = \underset{\sim}{R}_o(\underset{\sim}{I} + \underset{\sim}{\Omega}) \quad \text{as} \quad \varepsilon_o \to 0 \ , \tag{4.41}$$

subject to the condition that $(\underset{\sim}{I} + \underset{\sim}{\Omega}) = \underset{\sim}{I}$ at $\underset{\sim}{X} = {}_o\underset{\sim}{X}$ or equivalently

$$\underset{\sim}{\Omega} = \underset{\sim}{0} \ , \quad \underset{\sim}{R} = \underset{\sim}{R}_o \quad \text{at} \quad \underset{\sim}{X} = {}_o\underset{\sim}{X} \ , \tag{4.42}$$

where $\underset{\sim}{\Omega} = \underset{\sim}{0}(\varepsilon_o)$ as $\varepsilon_o \to 0$ is defined in (4.35). This completes the proof.

It may be observed that in the present development, apart from rigid body rotation, the conclusion $\underset{\sim}{\Omega} = \underset{\sim}{R} - \underset{\sim}{I} = \underset{\sim}{0}(\varepsilon_o)$ as $\varepsilon_o \to 0$ is a derived result. This is in contrast to the usual approach to infinitesimal kinematics, where the infinitesimal nature of $\underset{\sim}{\Omega}$ (and also of $\underset{\sim}{E}$) is implied through an assumption of smallness imposed on the displacement gradient $\underset{\sim}{H}$.

The restrictions in the statement of Theorem 4.2 are imposed on $\underset{\sim}{E}$ and $\underset{\sim}{E}_{,K}$ (instead of on the usual displacement gradients) and for an explicit calculation of the rotation tensor we need to return to (3.14). Alternatively, in view of the representation (4.41), it will suffice to calculate the skew-symmetric tensor $\underset{\sim}{\Omega}$ and then $\underset{\sim}{R}$ is determined. To see this, introduce (4.39) into (4.13) and after neglect ot terms of $\underset{\sim}{0}(\varepsilon_o^2)$ as $\varepsilon_o \to 0$, obtain

$$\underset{\sim}{A}_K = -\underset{\sim}{A}_K^T = \underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L) - (\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_{,L} = \underset{\sim}{O}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \quad . \tag{4.43}$$

But the expression for the gradient of $\underset{\sim}{R}$ by (3.14) is $\underset{\sim}{R}_{,K} = \underset{\sim}{R}\,\underset{\sim}{A}_K$ and substitution from (4.41) results in

$$\underset{\sim}{R}_o \underset{\sim}{\Omega}_{,K} = \underset{\sim}{R}_o(\underset{\sim}{I} + \underset{\sim}{\Omega})\underset{\sim}{A}_K \quad . \tag{4.44}$$

Since $\underset{\sim}{R}_o$ is nonsingular and since $\underset{\sim}{\Omega} = \underset{\sim}{O}(\varepsilon_o)$ and $\underset{\sim}{A}_K = \underset{\sim}{O}(\varepsilon_o)$ as $\varepsilon_o \to 0$ by $(4.35)_2$ and (4.43), respectively, (4.44) gives*

$$\underset{\sim}{\Omega}_{,K} = \underset{\sim}{A}_K \quad , \tag{4.45}$$

where terms of $\underset{\sim}{O}(\varepsilon_o^2)$ have been neglected. The tensor $\underset{\sim}{\Omega}$ can now be determined as a solution of (4.45) subject to the condition (4.42).

With the use of (2.1), $(4.11)_2$ and (4.41) we note that the deformation gradient is now given by

$$\underset{\sim}{F} = \underset{\sim}{R}_o(\underset{\sim}{I} + \underset{\sim}{\Omega} + \underset{\sim}{E}) \quad \text{as} \quad \varepsilon_o \to 0 \quad , \tag{4.46}$$

where terms of $\underset{\sim}{O}(\varepsilon_o^2)$ have been neglected. In addition, by (2.6) and (4.46) we also have

$$\underset{\sim}{H} = (\underset{\sim}{R}_o - \underset{\sim}{I}) + \underset{\sim}{R}_o(\underset{\sim}{\Omega} + \underset{\sim}{E}) \quad \text{as} \quad \varepsilon_o \to 0 \quad . \tag{4.47}$$

In the above expression, it is $\underset{\sim}{H} - (\underset{\sim}{R}_o - \underset{\sim}{I})$ and not $\underset{\sim}{H}$ itself which is infinitesimal with respect to ε_o.

4.3 Small strain accompanied by moderate rotation

The foregoing development [between (4.32) and (4.47)] which began with Definition 4.1 dealt with infinitesimal kinematics. We now return to our main objective and introduce

Definition 4.2. Given $\underset{\sim}{E} = \underset{\sim}{O}(\varepsilon_o)$, a proper orthogonal tensor $\underset{\sim}{R}$ is

*The component form of Eqs. (4.45) are, of course, the same as those used in the infinitesimal theory of motion mentioned in the first paragraph of section 1.

said to be a moderate rotation with respect to ε_o if for any unit vector $\underset{\sim}{v}$, the vector $\underset{\sim}{\beta}$ defined in (4.29) satisfies

$$\underset{\sim}{\beta} = \underset{\sim}{O}(\varepsilon_o^{1/2}) \quad \text{as} \quad \varepsilon_o \to 0 \ . \tag{4.48}$$

It follows at once from (4.48) and $(4.30)_2$ that $\underset{\sim}{R}$ is a moderate rotation in the sense of (4.48) if and only if [compare with (4.33)]

$$1 - \underset{\sim}{v} \cdot \underset{\sim}{R}\,\underset{\sim}{v} = O(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \ , \tag{4.49}$$

i.e., the projection of $\underset{\sim}{R}\,\underset{\sim}{v}$ along $\underset{\sim}{v}$ differs from unity by $O(\varepsilon_o)$ as $\varepsilon_o \to 0$ in the case of moderate rotation.

Observing from $(4.30)_3$ and (4.48) that for moderate rotation: $\cos\alpha = 1 + O(\varepsilon_o)$ as $\varepsilon_o \to 0$, the inequality (4.28) together with the fact that $\alpha = \theta$ for some $\underset{\sim}{v}$ imply that

$$\cos\theta = 1 + O(\varepsilon_o) \ , \ \sin^2\theta = O(\varepsilon_o) \ , \ \sin\theta = O(\varepsilon_o^{1/2}) \ , \ \theta = O(\varepsilon_o^{1/2}) \quad \text{as} \quad \varepsilon_o \to 0 \ . \tag{4.50}$$

Clearly, (4.49) can be used to state an alternative definition of moderate rotation which is equivalent to (4.48).

As in the Definition 4.1, we again observe that implicit in the Definition 4.2 is the assumption that the order of magnitude of $\underset{\sim}{R}$ can be estimated in terms of $\underset{\sim}{E}$. Again to avoid undue complications, we have postponed making explicit any relationship between ε_1 and ε_o but the conditions under which moderate rotation can be estimated in terms of infinitesimal strain will be examined later. However, two aspects of the conclusions (4.50) for moderate rotation may be noted here: (1) The results (4.50) have been obtained with respect to ε_o and are independent of ε_1 defined by (4.2); and (2) the angle of rotation in (4.50) is of $O(\varepsilon_o^{1/2})$ in constrast to that of $O(\varepsilon_o)$ in (4.34) for infinitesimal rotation. Thus, after substituting from $(4.50)_{1,3}$ in $(2.14)_{1,2}$ and

neglecting terms of $O(\varepsilon_o^{3/2})$, the functions $\underset{\sim}{\Psi}$ and $\underset{\sim}{\Phi}$ in (2.13) for moderate rotation are approximated according to

$$\begin{aligned} &\underset{\sim}{\Psi} = -\underset{\sim}{\Psi}^T = \underset{\sim}{O}(\varepsilon_o^{1/2}) \quad , \quad \underset{\sim}{\Phi} = \underset{\sim}{\Phi}^T = \underset{\sim}{O}(\varepsilon_o) \quad , \\ &\underset{\sim}{\Psi}\,\underset{\sim}{\Psi} = 2\underset{\sim}{\Phi} = \underset{\sim}{O}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \quad . \end{aligned} \tag{4.51}$$

It should be noted here that if $\underset{\sim}{R}$ is a moderate rotation with respect to ε_o everywhere in the body, then in terms of the condition (4.7) with $m=2$, $n=0$, $k=2$, $\ell=1$, quantities of $O(\varepsilon_2)$ are also of $O(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$.

We now state the following

<u>Theorem 4.3</u>. If $\underset{\sim}{E} = \underset{\sim}{U} - \underset{\sim}{I} = \underset{\sim}{O}(\varepsilon_o)$ as $\varepsilon_o \to 0$ and $\underset{\sim}{R}$ is a moderate rotation in the senso of Definition 4.2, then the tensor $\underset{\sim}{H}$ defined by (2.6) is of $\underset{\sim}{O}(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$. Conversely, if $\underset{\sim}{H} = \underset{\sim}{O}(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$ and, in addition, if the symmetric part of (2.6), i.e., $\underset{\sim}{H} + \underset{\sim}{H}^T$ is of $\underset{\sim}{O}(\varepsilon_o)$ as $\varepsilon_o \to 0$, then $\underset{\sim}{U} - \underset{\sim}{I} = O(\varepsilon_o)$ and $\underset{\sim}{R}$ is a moderate rotation with respect to ε_o.

<u>Proof</u>. We first prove the first part of the theorem. If $\underset{\sim}{U} - \underset{\sim}{I} = \underset{\sim}{E} = \underset{\sim}{O}(\varepsilon_o)$ as $\varepsilon_o \to 0$ and $\underset{\sim}{R}$ is a moderate rotation given by (2.13) with $\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}$ estimated by (4.51), then recalling (2.1) we have

$$\begin{aligned} &\underset{\sim}{F} = \underset{\sim}{I} + \underset{\sim}{\Psi} + \underset{\sim}{\Phi} + \underset{\sim}{E} \quad , \quad \underset{\sim}{E} = \underset{\sim}{O}(\varepsilon_o) \quad , \\ &\underset{\sim}{\Phi}, \underset{\sim}{\Psi} \text{ given by (4.51)} \quad \text{as} \quad \varepsilon_o \to 0 \end{aligned} \tag{4.52}$$

and $\underset{\sim}{H} = \underset{\sim}{O}(\varepsilon_o^{1/2})$ by (2.6).

We now turn to the converse part of the theorem. By assumption, $\underset{\sim}{H} = \underset{\sim}{O}(\varepsilon_o^{1/2})$ and $\underset{\sim}{H} + \underset{\sim}{H}^T = O(\varepsilon_o)$ as $\varepsilon_o \to 0$. Hence, substitution into $(2.7)_1$ yields $\underset{\sim}{E} = \underset{\sim}{O}(\varepsilon_o)$ as $\varepsilon_o \to 0$ and by $(4.11)_2$ we also have $\underset{\sim}{U} = \underset{\sim}{I} + \underset{\sim}{O}(\varepsilon_o)$. Further, in order to obtain the desired estimate for $\underset{\sim}{R}$, from (2.1) we have $\underset{\sim}{R} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1}$ and from $(4.11)_3$ we recall that

$$\underset{\sim}{U}^{-1} = \underset{\sim}{I} - \underset{\sim}{E} = \underset{\sim}{I} - \tfrac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T + \underset{\sim}{H}^T\underset{\sim}{H}) \quad \text{as} \quad \varepsilon_o \to 0 \quad . \tag{4.53}$$

Then, $\underset{\sim}{R}$ can be expressed as

$$\underset{\sim}{R} = (\underset{\sim}{I} + \underset{\sim}{H})\{\underset{\sim}{I} - \tfrac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T + \underset{\sim}{H}^T\underset{\sim}{H})\}$$

$$= \underset{\sim}{I} + \tfrac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T) - \tfrac{1}{2}\underset{\sim}{H}^T\underset{\sim}{H} + \underset{\sim}{O}(\varepsilon_o^{3/2}) \quad . \tag{4.54}$$

A close examination of (4.54) reveals that if we identify the skew-symmetric $\frac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T)$ as $\underset{\sim}{\Psi}$ and the symmetric $-\frac{1}{2}\underset{\sim}{H}^T\underset{\sim}{H}$ as $\underset{\sim}{\Phi}$ so that these quantities can meet the condition $(4.51)_3$, then $\underset{\sim}{R}$ in (4.54) can be identified as moderate rotation. To show this, we recall that by assumption

$$\underset{\sim}{H} = -\underset{\sim}{H}^T + \underset{\sim}{O}(\varepsilon_o) \quad \text{and} \quad \underset{\sim}{H}^T = -\underset{\sim}{H} + \underset{\sim}{O}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \quad . \tag{4.55}$$

Thus, after forming the product $\underset{\sim}{\Psi}\,\underset{\sim}{\Psi} = [\frac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T)][\frac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T)]$ and neglecting terms of $\underset{\sim}{O}(\varepsilon_o^{3/2})$, we obtain

$$[\tfrac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T)][\tfrac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T)] = \tfrac{1}{4}\{-2\underset{\sim}{H}^T + \underset{\sim}{O}(\varepsilon_o)\}\{2\underset{\sim}{H} + \underset{\sim}{O}(\varepsilon_o)\}$$

$$= -\underset{\sim}{H}^T\underset{\sim}{H} \quad , \tag{4.56}$$

which meets $(4.51)_3$. Hence, $\underset{\sim}{R}$ given by (4.54) is a moderate rotation and the proof of the theorem is complete.

We recall once more that the strain gradients $\underset{\sim}{E}_{,K} = \underset{\sim}{O}(\varepsilon_1)$ as $\varepsilon_1 \to 0$, in view of (4.2), and that the estimate (4.23) for the angle of rotation θ is also obtained with respect to ε_1. This estimate (4.23) can be brought into correspondence with $(4.50)_4$, which results from Definition 4.2, if and only if functions of $O(\varepsilon_1)$ are assumed to be comparable to $O(\varepsilon_o^{1/2})$. In terms of the conditions stated following (4.4), we may choose $k = 2$, $\ell = 1$ so that any function of $O(\varepsilon_1) = O(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$. With this additional assumption, we may now write

$$\underset{\sim}{E}_{,K} = \underset{\sim}{O}(\varepsilon_o^{1/2}) \quad \text{as} \quad \varepsilon_o \to 0 \tag{4.57}$$

and this is consistent with

$$\theta = O(\varepsilon_o^{1/2}) \quad \text{as} \quad \varepsilon_o \to 0 \quad , \tag{4.58}$$

provided that $\underset{\sim}{R} = \underset{\sim}{I}$ at some point ${}_o\underset{\sim}{X} \in \mathscr{B}$.

We now state a theorem concerning the order of magnitude of $\underset{\sim}{R}$:

Theorem 4.4. Given $\underset{\sim}{E} = \underset{\sim}{O}(\varepsilon_o)$ and Grad $\underset{\sim}{E} = \underset{\sim}{O}(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$, the tensor $\underset{\sim}{R}$ associated with the deformation function $\underset{\sim}{\chi}$ is a moderate rotation with respect to ε_o to within a rigid body rotation.

Proof. Provided that $\underset{\sim}{R} = \underset{\sim}{I}$ at some point ${}_o\underset{\sim}{X} \in \mathscr{B}$, the conclusion (4.58) implies that $\underset{\sim}{R}$ is a moderate rotation with respect to ε_o in the sense of Definition 4.2. Thus, with the use of (2.13) and (4.51) and to within terms of $O(\varepsilon_o^{3/2})$, the moderate rotation is given by

$$\left.\begin{array}{c} \underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi} \quad \text{as} \quad \varepsilon_o \to 0 \quad , \\ \underset{\sim}{\Phi} \text{ and } \underset{\sim}{\Psi} \text{ specified by } (4.51)_{1,2} \quad . \end{array}\right\} \tag{4.59}$$

Moreover, by Theorem 3.1 any other solution of (3.14) differs from (4.59) by a proper orthogonal tensor function of time (say $\underset{\sim}{R}_o$) corresponding to a rigid body rotation and we have

$$\left.\begin{array}{c} \underset{\sim}{R} = \underset{\sim}{R}_o(\underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi}) \quad , \\ \underset{\sim}{\Phi} \text{ and } \underset{\sim}{\Psi} \text{ specified by } (4.51)_{1,2} \quad , \end{array}\right\} \tag{4.60}$$

subject to the conditions that

$$\underset{\sim}{\Phi} = \underset{\sim}{\Psi} = \underset{\sim}{0} \quad , \quad \underset{\sim}{R} = \underset{\sim}{R}_o \quad \text{at} \quad \underset{\sim}{X} = {}_o\underset{\sim}{X} \tag{4.61}$$

and the theorem is proved.

It should be emphasized here that the approximations for moderate rotation occur only in the functions $\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}$ as specified by $(4.51)_{1,2}$.

The rotation $\underset{\sim}{R}$ in (4.60) may in fact be large, in view of the presence of $\underset{\sim}{R}_o$ which represents a rigid body rotation. Indeed, it is the proper orthogonal quantity $\underset{\sim}{R}_o^T\underset{\sim}{R}$ which may be termed moderate rotation and not $\underset{\sim}{R}$.

The restrictions in the statement of Theorem 4.4 are imposed on $\underset{\sim}{E}$ and $\underset{\sim}{E}_{,K}$ and for an explicit calculation of the rotation tensor we need to return to (3.14). Alternatively, in view of the representation (4.60), we may calculate $\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}$ to the order of approximation considered and then $\underset{\sim}{R}$ is determined. To show this, introduce (4.57) into (4.13) and after neglect of terms of $\underset{\sim}{0}(\varepsilon_o^{3/2})$ as $\varepsilon_o \to 0$, obtain

$$\underset{\sim}{A}_K = -\underset{\sim}{A}_K^T = \underset{\sim}{E}_{,L}(\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L) - (\underset{\sim}{e}_L \otimes \underset{\sim}{e}_K)\underset{\sim}{E}_{,L} = \underset{\sim}{0}(\varepsilon_o^{\frac{1}{2}}) \quad \text{as} \quad \varepsilon_o \to 0 \ . \tag{4.62}$$

Substitution of (4.60) into $\underset{\sim}{R}_{,K} = \underset{\sim}{R}\,\underset{\sim}{A}_K$, which is obtained from (3.14), results in

$$\underset{\sim}{R}_o(\underset{\sim}{\Phi}_{,K} + \underset{\sim}{\Psi}_{,K}) = \underset{\sim}{R}_o(\underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi})\underset{\sim}{A}_K \ . \tag{4.63}$$

But, with the use of the order of magnitude estimates $(4.51)_{1,2}$ and (4.62), after neglecting terms of $\underset{\sim}{0}(\varepsilon_o^{3/2})$ and remembering that $\underset{\sim}{R}_o$ is nonsingular, from (4.63) we obtain

$$\underset{\sim}{\Phi}_{,K} + \underset{\sim}{\Psi}_{,K} = \underset{\sim}{A}_K + \underset{\sim}{\Psi}\,\underset{\sim}{A}_K \ . \tag{4.64}$$

Since $\underset{\sim}{\Psi}_{,K}$ represents the skew-symmetric part of (4.64), we may write

$$\begin{aligned} \underset{\sim}{\Psi}_{,K} &= \tfrac{1}{2}\{(\underset{\sim}{\Phi}_{,K} + \underset{\sim}{\Psi}_{,K}) - (\underset{\sim}{\Phi}_{,K} + \underset{\sim}{\Psi}_{,K})^T\} \\ &= \tfrac{1}{2}\{(\underset{\sim}{A}_K + \underset{\sim}{\Psi}\,\underset{\sim}{A}_K) - (\underset{\sim}{A}_K + \underset{\sim}{\Psi}\,\underset{\sim}{A}_K)^T\} \\ &= \underset{\sim}{A}_K + \tfrac{1}{2}(\underset{\sim}{\Psi}\,\underset{\sim}{A}_K - \underset{\sim}{A}_K\underset{\sim}{\Psi}) \ , \end{aligned} \tag{4.65}$$

where $(4.62)_1$ has been used in obtaining $(4.65)_3$. Given $\underset{\sim}{E}_{,K}$ (and hence $\underset{\sim}{A}_K$), $(4.65)_3$ may be viewed as the differential equation for $\underset{\sim}{\Psi}$; and, once $\underset{\sim}{\Psi}$ is

determined, $\underset{\sim}{\Phi}$ can then be found from $(4.51)_3$.

It is useful to record here the expressions for the deformation gradient $\underset{\sim}{F}$ and the relative deformation gradient $\underset{\sim}{H}$ associated with moderate rotation. Thus, with the use of (2.1), $(4.11)_2$, (4.60) and upon neglect of terms of $\underset{\sim}{O}(\varepsilon_o^{3/2})$, the deformation gradient can be expressed as

$$\underset{\sim}{F} = \underset{\sim}{R}_o(\underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi} + \underset{\sim}{E}) \quad \text{as} \quad \varepsilon_o \to 0 \ . \tag{4.66}$$

Similarly, by (2.6) and (4.66), we have

$$\underset{\sim}{H} = (\underset{\sim}{R}_o - \underset{\sim}{I}) + \underset{\sim}{R}_o(\underset{\sim}{\Psi} + \underset{\sim}{\Phi} + \underset{\sim}{E}) \ . \tag{4.67}$$

In view of the remarks made in the paragraph following (4.61), it should be noted that in the above expression it is $[\underset{\sim}{H} - (\underset{\sim}{R}_o - \underset{\sim}{I})]$ and not $\underset{\sim}{H}$ itself which is moderately large.

4.4 Construction of a motion which results in a properly invariant small strain and moderate rotation.

From among all particles of $\mathcal{B}$, let Y be chosen as a pivot whose translation and rotation is specified (Casey and Naghdi 1981). Then, corresponding to any motion $\underset{\sim}{\chi}$, we can construct another motion $\underset{\sim}{\chi}^*$ by removing from $\underset{\sim}{\chi}$ the translation and rotation of the pivot. Thus, we can write

$$\begin{aligned} \underset{\sim}{x}^* = \underset{\sim}{\chi}^*(\underset{\sim}{X}, t^*) &= \underset{\sim}{R}_o^T\{\underset{\sim}{\chi}(\underset{\sim}{X}, t) - \underset{\sim}{\chi}({}_o\underset{\sim}{X}, t)\} + {}_o\underset{\sim}{X} \ , \\ t^* &= t - c \ , \end{aligned} \tag{4.68}$$

where ${}_o\underset{\sim}{X}$ denotes the position vector of the pivot Y in the reference configuration $\underset{\sim}{\kappa}_o$, $\underset{\sim}{R}_o = \underset{\sim}{R}({}_o\underset{\sim}{X}, t)$ is only a function of time and c is a real constant. The configuration of $\mathcal{B}$ at time $\underset{\sim}{t}^*$ in the motion $\underset{\sim}{\chi}^*$ is $\underset{\sim}{\kappa}^*$. We observe that (4.68) is of the form (2.9) with

$$\underset{\sim}{Q}(t) = \underset{\sim}{R}_o^T \ , \quad \underset{\sim}{a}(t) = -\underset{\sim}{R}_o^T\underset{\sim}{\chi}({}_o\underset{\sim}{X}, t) + {}_o\underset{\sim}{X} \ , \quad a = -c \ . \tag{4.69}$$

Then, with the use of (2.10), (2.11), (4.60) and (4.66), we easily conclude that

$$\underset{\sim}{F}^{*} = \underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi} + \underset{\sim}{E} \quad \text{as} \quad \varepsilon_o \to 0 \ ,$$
$$\underset{\sim}{R}^{*} = \underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi} \quad \text{as} \quad \varepsilon_o \to 0 \ , \tag{4.70}$$

and that

$$\underset{\sim}{C}^{*} = \underset{\sim}{C} \ , \quad \underset{\sim}{U}^{*} = \underset{\sim}{U} \ , \quad \underset{\sim}{E}^{*} = \underset{\sim}{E} \ , \quad \underset{\sim}{E}^{*}_{,K} = \underset{\sim}{E}_{,K} \ . \tag{4.71}$$

In the configuration κ^*, the relative displacement $\underset{\sim}{u}^* = \underset{\sim}{\chi}^* - \underset{\sim}{X}$, and the rotation tensor $\underset{\sim}{R}^* = \underset{\sim}{R}_o^T \underset{\sim}{R}$ are such that

$$\underset{\sim}{u}^{*}({}_o\underset{\sim}{X}, t^*) = \underset{\sim}{0} \ , \quad \underset{\sim}{R}^{*}({}_o\underset{\sim}{X}, t^*) = \underset{\sim}{I} \ . \tag{4.72}$$

The expressions for Grad $\underset{\sim}{u}^* = \underset{\sim}{u}^*_{,K} \otimes \underset{\sim}{e}_K$ are again of the form (2.6) but with $\underset{\sim}{H}$ replaced by

$$\underset{\sim}{H}^{*} = \underset{\sim}{u}^{*}_{,K} \otimes \underset{\sim}{e}_{,K} = \underset{\sim}{\Phi} + \underset{\sim}{\Psi} + \underset{\sim}{E} \ . \tag{4.73}$$

It is clear from $(4.51)_{1,2}$, $(4.11)_1$ and the right-hand side ot $(4.73)_2$ that $\underset{\sim}{u}^*_{,K} = 0(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$. From this result and the fact that $\underset{\sim}{u}^*$ vanishes at ${}_o\underset{\sim}{X}$ by $(4.72)_1$, it follows that the displacement

$$\underset{\sim}{u}^{*}(\underset{\sim}{X}, t^*) = \underset{\sim}{0}(\varepsilon_o^{1/2}) \quad \text{as} \quad \varepsilon_o \to 0 \tag{4.74}$$

throughout the body[+]. Similar to $(2.7)_1$, the strain measure $\underset{\sim}{E}^*$ can be calculated in terms of $\underset{\sim}{H}^*$ so that

$$\begin{aligned} \underset{\sim}{E}^{*} &= \tfrac{1}{2}(\underset{\sim}{H}^{*} + \underset{\sim}{H}^{*T} + \underset{\sim}{H}^{*T}\underset{\sim}{H}) \\ &= \tfrac{1}{2}\{\underset{\sim}{u}^{*}_{,K} \otimes \underset{\sim}{e}_K + \underset{\sim}{e}_K \otimes \underset{\sim}{u}^{*}_{,K} + (\underset{\sim}{u}^{*}_{,K} \cdot \underset{\sim}{u}^{*}_{,L})\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L\} \ . \end{aligned} \tag{4.75}$$

It should be noted that in the above formulae for infinitesimal strain,

---

[+]The line of argument here is similar to that employed between (4.20)-(4.23).

while both quantities $\underset{\sim}{H}^* + \underset{\sim}{H}^{*T}$ and $\underset{\sim}{H}^{*T}\underset{\sim}{H}^*$ are of $\underset{\sim}{O}(\varepsilon_o)$, $\underset{\sim}{H}^* = \underset{\sim}{O}(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$.

It has been shown in the paper of Casey and Naghdi (1981, Theorem 3.1) that two motions ${}_1\underset{\sim}{\chi}$ and ${}_2\underset{\sim}{\chi}$ of $\mathcal{B}$ differ by a rigid motion if and only if ${}_1\underset{\sim}{\chi}^* = {}_2\underset{\sim}{\chi}^*$, i.e., by construction $\underset{\sim}{\chi}^*$ remains unchanged if $\underset{\sim}{\chi}$ is replaced by $\underset{\sim}{\chi}^+$ in (2.9). Thus, with $(\underset{\sim}{\chi}^+)^*$ defined by [see section 3.2 of Casey and Naghdi (1981)]

$$(\underset{\sim}{\chi}^+)^*(\underset{\sim}{X},t) = (\underset{\sim}{R}_o^+)^T\{\underset{\sim}{\chi}^+(\underset{\sim}{X},t^+) - \underset{\sim}{\chi}^+({}_o\underset{\sim}{X},t^+)\} + {}_o\underset{\sim}{X} \tag{4.76}$$

and with c chosen equal to -a in (2.9), all quantities with respect to the configuration $\underset{\sim}{\kappa}^*$ remain unaffected by an arbitrary (not necessarily small) superposed rigid body motions and we conclude that

$$\begin{gathered}(\underset{\sim}{\chi}^+)^* = \underset{\sim}{\chi}^* \ , \quad \underset{\sim}{F}^{+*} = \underset{\sim}{F}^* = \underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi} + \underset{\sim}{E} \ , \\ \underset{\sim}{R}^{+*} = \underset{\sim}{R}^* = \underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi} \ , \quad \underset{\sim}{H}^{+*} = \underset{\sim}{H}^* = \underset{\sim}{\Phi} + \underset{\sim}{\Psi} + \underset{\sim}{E} \ , \\ \underset{\sim}{C}^{+*} = \underset{\sim}{C}^* \ , \quad \underset{\sim}{U}^{+*} = \underset{\sim}{U} \ , \quad \underset{\sim}{E}^{+*} = \underset{\sim}{E}^* \ , \quad \underset{\sim}{E}^{+*}_{,K} = \underset{\sim}{E}^*_{,K} \ .\end{gathered} \tag{4.77}$$

where $(\underset{\sim}{F}^+)^* = \mathrm{Grad}(\underset{\sim}{\chi}^+)^*$.

5. Invariance of constitutive equations with small strain accompanied by moderate rotation

We briefly discuss here the invariance of constitutive equations (and hence that of a complete theory) for small strain accompanied by moderate rotation. Although the development of this section is specifically carried out for the case of an elastic material, it will be clear that our main conclusion reached holds for any material. The procedure for constructing a properly invariant theory in the presence of moderate rotation is similar to that used by Casey and Naghdi (1981) for an infinitesimal theory of motion in which, apart from superposed rigid body motions, both strain and rotation are small.

The notations for the mass density, in the configuration $\underset{\sim}{\kappa}$, the outward unit normal to the surface $\partial\mathcal{R}$, the stress vector $\underset{\sim}{t}$ acting on $\partial\mathcal{R}$ and the Cauchy stress tensor $\underset{\sim}{T}$ were introduced in section 2. We denote the corresponding quantities in the motion $\underset{\sim}{\chi}^*$, introduced in (4.68), and in the configuration $\underset{\sim}{\kappa}^*$ by $\rho^*, \underset{\sim}{n}^*, \underset{\sim}{t}^*$ and $\underset{\sim}{T}^*$, respectively. Recalling that $\underset{\sim}{R}^T({}_o\underset{\sim}{X},t) = \underset{\sim}{R}_o^T$ in (4.68) is a function of time only and plays the role of $\underset{\sim}{Q}(t)$ in (2.9), as noted also in $(4.69)_1$, it follows from $(2.16)_1$, $(2.16)_2$, (2.17) and (2.18) that

$$\begin{gathered}\rho^* = \rho \quad , \quad \underset{\sim}{n}^* = \underset{\sim}{R}^T({}_o\underset{\sim}{X},t)\underset{\sim}{n} \quad , \quad \underset{\sim}{t}^* = \underset{\sim}{R}^T({}_o\underset{\sim}{X},t)\underset{\sim}{t} \quad , \\ \underset{\sim}{T}^* = \underset{\sim}{R}^T({}_o\underset{\sim}{X},t)\underset{\sim}{T}\,\underset{\sim}{R}\,({}_o\underset{\sim}{X},t) \quad .\end{gathered} \tag{5.1}$$

Similarly, associated with the motion $(\underset{\sim}{\chi}^+)^*$ defined by (4.67), we have the quantities ρ^{+*}, $\underset{\sim}{n}^{+*}$, $\underset{\sim}{t}^{+*}$ and $\underset{\sim}{T}^{+*}$. These quantities, with the help of $(5.1)_{1,2,3,4}$, (2.11) and (2.16)-(2.18) transform according to (Casey and Naghdi 1981, section 3):

$$\rho^{+*} = \rho^* \quad , \quad \underset{\sim}{n}^{+*} = \underset{\sim}{n}^* \quad , \quad \underset{\sim}{t}^{+*} = \underset{\sim}{t}^* \quad ,$$

$$\underset{\sim}{T}^{+*} = \{\underset{\sim}{Q}(t)\underset{\sim}{R}_o\}^T \underset{\sim}{Q}(t)\underset{\sim}{T}\,\underset{\sim}{Q}^T(t)\{\underset{\sim}{Q}(t)\underset{\sim}{R}_o\} = \underset{\sim}{T}^* \quad . \tag{5.2}$$

For an elastic material, let $\psi = \psi(\underset{\sim}{X},t)$ denote the elastic strain energy per unit mass in the configuration $\underset{\sim}{\kappa}$. Also, let ψ^+ and ψ^* denote the strain energy per unit mass in the configurations $\underset{\sim}{\kappa}^+$ and $\underset{\sim}{\kappa}^*$, respectively. We assume that $\psi^+ = \psi$ and it then follows that $\psi^* = \psi$. The nonlinear behavior of an elastic solid may be characterized by the constitutive equation

$$\underset{\sim}{T} = \tfrac{1}{2}\rho\underset{\sim}{F}\{\frac{\partial\hat{\psi}}{\partial\underset{\sim}{E}} + (\frac{\partial\hat{\psi}}{\partial\underset{\sim}{E}})^T\}\underset{\sim}{F}^T \quad , \tag{5.3}$$

where $\psi = \hat{\psi}(\underset{\sim}{E})$. We observe that in view of (2.10), $(2.16)_1$ and the fact that $\underset{\sim}{E}^+ = \underset{\sim}{E}$ the value of $\underset{\sim}{T}^+$ of the stress tensor given by (5.3) for the motion $\underset{\sim}{\chi}^+$ satisfies (2.18), so that (5.3) is a properly invariant constitutive equation. The Cauchy stress $\underset{\sim}{T}^*$ in the motion $\underset{\sim}{\chi}^*$ has the form

$$\underset{\sim}{T}^* = \tfrac{1}{2}\rho^*\underset{\sim}{F}^*\{\frac{\partial\hat{\psi}}{\partial\underset{\sim}{E}^*} + (\frac{\partial\hat{\psi}}{\partial\underset{\sim}{E}^*})^T\}(\underset{\sim}{F}^*)^T \tag{5.4}$$

and satisfies $(5.2)_4$.

Suppose now that the motion is such that $\underset{\sim}{E} = \underset{\sim}{0}(\varepsilon_o)$ and $\underset{\sim}{E}_{,K} = \underset{\sim}{0}(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$ and recall that for such a motion the deformation gradient is given by (4.66). It then follows from the local equation for conservation of mass, namely $\rho \det \underset{\sim}{F} = \rho_o$, that for small strain ρ can be approximated as

$$\rho = \rho_o + 0(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \quad , \tag{5.5}$$

where ρ_o is the mass density in the reference configuration $\underset{\sim}{\kappa}_o$. Further, assume for simplicity that the strain energy $\hat{\psi}$ is quadratic in $\underset{\sim}{E}$ so that

$$\tfrac{1}{2}\{\frac{\partial\hat{\psi}}{\partial \underset{\sim}{E}} + (\frac{\partial\hat{\psi}}{\partial \underset{\sim}{E}})^T\} = \underset{\sim}{\mathcal{K}}[\underset{\sim}{E}] \quad , \tag{5.6}$$

where $\underset{\sim}{\mathcal{K}}$ is a constant fourth order tensor and $\{\underset{\sim}{\mathcal{K}}[\underset{\sim}{E}]\}^T = \underset{\sim}{\mathcal{K}}[\underset{\sim}{E}]$. With the help of (5.6), (4.66), $(4.11)_1$, (4.51), (5.5) and after neglecting terms of $\underset{\sim}{O}(\varepsilon_o^{3/2})$ as $\varepsilon_o \to 0$, from (5.3) we obtain

$$\underset{\sim}{T} = \rho_o \underset{\sim}{R}_o \underset{\sim}{\mathcal{K}}[\underset{\sim}{E}] \underset{\sim}{R}_o^T \quad . \tag{5.7}$$

After substituting (5.7) into $(5.1)_4$, we have

$$\underset{\sim}{T}^* = \underset{\sim}{R}_o^T \underset{\sim}{T} \underset{\sim}{R}_o = \rho_o \underset{\sim}{\mathcal{K}}[\underset{\sim}{E}] \quad . \tag{5.8}$$

It is now clear that the constitutive equation (5.8) meets the requirement $(5.2)_4$ and is properly invariant. As noted in subsection 4.4, the method of construction of Casey and Naghdi (1981), which is also used in the development of this section, removes from all motions the translation and rotation at any particle Y of the body called a pivot. But, even in the presence of moderate rotation, it can be demonstrated that it does not matter which particle is chosen as pivot. Indeed, it is shown in Appendix C that the theory in which the deformation gradient given by (4.66) is constructed with Y' as pivot coincides, to within terms of $\underset{\sim}{O}(\varepsilon_o^{3/2})$, with that having Y as pivot.

Appendix A

A brief account of notation and mathematical terminology used in the paper is given at the end of section 1. In this Appendix we collect additional terminologies and mathematical results utilized in the development of the paper.

Any linear mapping between a set of vectors and a set of second order tensors will be regarded as a third order tensor. In particular, the tensor product $\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c}$ of any three vectors $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c} \in V$ is a third order tensor defined by $(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})\underset{\sim}{v} = \underset{\sim}{c} \cdot \underset{\sim}{v}\, \underset{\sim}{a} \otimes \underset{\sim}{b}$, and we also define a product $(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})[\underset{\sim}{u} \otimes \underset{\sim}{v}] = (\underset{\sim}{b} \cdot \underset{\sim}{u})(\underset{\sim}{c} \cdot \underset{\sim}{v})\underset{\sim}{a}$, for any vector $\underset{\sim}{u},\underset{\sim}{v} \in V$. We note that the tensor product between a second order tensor and a vector, i.e., $\underset{\sim}{T} \otimes \underset{\sim}{v}$ or $\underset{\sim}{v} \otimes \underset{\sim}{T}$ for any second order tensor $\underset{\sim}{T}$ and any vector $\underset{\sim}{v}$, is also defined as a third order tensor. The product of a third order tensor and a second order tensor is again a third order tensor defined by

$$
\begin{aligned}
(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})(\underset{\sim}{u} \otimes \underset{\sim}{v}) &= \underset{\sim}{c} \cdot \underset{\sim}{u}(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{v}) = (\underset{\sim}{a} \otimes \underset{\sim}{c})(\underset{\sim}{u} \otimes \underset{\sim}{b} \otimes \underset{\sim}{v}) \quad , \\
(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})\underset{\sim}{T} &= \underset{\sim}{a} \otimes \underset{\sim}{b} \otimes (\underset{\sim}{T}^T\underset{\sim}{c}) \quad , \quad \underset{\sim}{T}(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c}) = (\underset{\sim}{T}\underset{\sim}{a}) \otimes \underset{\sim}{b} \otimes \underset{\sim}{c} \quad ,
\end{aligned}
\tag{A1}
$$

for any vectors $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{u},\underset{\sim}{v} \in V$ and any second order tensor $\underset{\sim}{T}$. Since the transpose of a third order tensor is not uniquely defined, we introduce three kinds of transpose distinguished, respectively, by superscript T_1,T_2,T_3. Thus

$$
\begin{aligned}
(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})^{T_1} &= \underset{\sim}{a} \otimes (\underset{\sim}{b} \otimes \underset{\sim}{c})^T = \underset{\sim}{a} \otimes \underset{\sim}{c} \otimes \underset{\sim}{b} \quad , \\
(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})^{T_2} &= (\underset{\sim}{a} \otimes \underset{\sim}{b})^T \otimes \underset{\sim}{c} = \underset{\sim}{b} \otimes \underset{\sim}{a} \otimes \underset{\sim}{c} \quad , \\
(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})^{T_3} &= \underset{\sim}{c} \otimes \underset{\sim}{b} \otimes \underset{\sim}{a}
\end{aligned}
\tag{A2}
$$

for any vector $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c} \in V$, and we note that the definition $(A2)_3$ involves

the transposition of $\underset{\sim}{c}$ and $\underset{\sim}{a}$. It is clear that as a consequence of these definitions, our definition for the transpose T_3 can be related to those for the transpose T_1 and T_2. Keeping this in mind, it is convenient to introduce the abbreviation $\underset{\sim}{\mathcal{A}}^{T_3} = [(\underset{\sim}{\mathcal{A}}^{T_1})^{T_2}]^{T_1} = [(\underset{\sim}{\mathcal{A}}^{T_2})^{T_1}]^{T_2}$ for a third order tensor $\underset{\sim}{\mathcal{A}}$.

The linear mapping from the set of second order tensors into itself is a fourth order tensor. In particular, the tensor product $\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c} \otimes \underset{\sim}{d}$ of any four vectors $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{d} \in V$ is a fourth order tensor. It is useful to record the relationship $(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c} \otimes \underset{\sim}{d})[\underset{\sim}{u} \otimes \underset{\sim}{v}] = (\underset{\sim}{c} \cdot \underset{\sim}{u})(\underset{\sim}{d} \cdot \underset{\sim}{v})\underset{\sim}{a} \otimes \underset{\sim}{b}$, which is a second order tensor. The transpose $\underset{\sim}{\mathcal{K}}^T$ of a fourth order tensor $\underset{\sim}{\mathcal{K}}$ is defined by the relationship $\underset{\sim}{B} \cdot \underset{\sim}{\mathcal{K}}[\underset{\sim}{A}] = \underset{\sim}{A} \cdot \underset{\sim}{\mathcal{K}}^T[\underset{\sim}{B}]$ for all second order tensors $\underset{\sim}{A},\underset{\sim}{B}$. Clearly, $(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c} \otimes \underset{\sim}{d})^T = (\underset{\sim}{c} \otimes \underset{\sim}{d} \otimes \underset{\sim}{a} \otimes \underset{\sim}{b})$.

We discussed the component representation of second order tensors in section 2. Similarly, for a third order tensor $\underset{\sim}{\mathcal{A}}$ or a fourth order tensor $\underset{\sim}{\mathcal{K}}$ we may write $\underset{\sim}{\mathcal{A}} = \mathcal{A}_{KLM}\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L \otimes \underset{\sim}{e}_M$ and $\underset{\sim}{\mathcal{K}} = \mathcal{K}_{KLMN}\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L \otimes \underset{\sim}{e}_M \otimes \underset{\sim}{e}_N$, where $\mathcal{A}_{KLM} = \underset{\sim}{e}_K \cdot \underset{\sim}{\mathcal{A}}[\underset{\sim}{e}_L \otimes \underset{\sim}{e}_M]$, and $\mathcal{K}_{KLMN} = (\underset{\sim}{e}_K \otimes \underset{\sim}{e}_L) \cdot \underset{\sim}{\mathcal{K}}[\underset{\sim}{e}_M \otimes \underset{\sim}{e}_N]$.

The gradient of a scalar-valued function $\phi(\underset{\sim}{X})$ may be written as

$$\text{Grad}\ \phi = \frac{\partial \phi}{\partial X_K} \underset{\sim}{e}_K \quad . \tag{A3}$$

Similarly, the gradient of a vector-valued function $\underset{\sim}{v}(\underset{\sim}{X})$ and the gradient of a second order tensor-valued function $\underset{\sim}{T}(\underset{\sim}{X})$ will be denoted by

$$\text{Grad}\ \underset{\sim}{v} = \frac{\partial v_L}{\partial X_K}\, \underset{\sim}{e}_L \otimes \underset{\sim}{e}_K \quad ,$$

$$\text{Grad}\ \underset{\sim}{T} = \frac{\partial T_{LM}}{\partial X_K}\, \underset{\sim}{e}_L \otimes \underset{\sim}{e}_M \otimes \underset{\sim}{e}_K = \underset{\sim}{T}_{,K} \otimes \underset{\sim}{e}_K \quad , \tag{A4}$$

while their components can be written as

$$(\text{Grad}\ \underset{\sim}{v})\underset{\sim}{e}_M = (\underset{\sim}{v}_{,K} \otimes \underset{\sim}{e}_K)\underset{\sim}{e}_M = \underset{\sim}{v}_{,K}\delta_{KM} = \underset{\sim}{v}_{,M} \quad ,$$

$$(\text{Grad}\ \underset{\sim}{T})\underset{\sim}{e}_M = (\underset{\sim}{T}_{,K} \otimes \underset{\sim}{e}_K)\underset{\sim}{e}_M = \underset{\sim}{T}_{,K}\delta_{KM} = \underset{\sim}{T}_{,M} \quad . \tag{A5}$$

Appendix B

This appendix provides details of the calculations of several results and formulae stated in sections 3 and 4. We begin by noting that the conditions (3.10) or (3.11) are regarded as the system of differential equations for the determination of $\underset{\sim}{F}$ from the knowledge of $\underset{\sim}{C}$. As such they were obtained as necessary conditions. To show that they are also sufficient, we consider the expression of the form

$$(F_{iL}F_{iM})_{,K} = F_{iL,K}F_{iM} + F_{iL}F_{iM,K} \quad . \tag{B1}$$

Assume that F_{iL} are the solutions of $(3.10)_2$. Then, by direct substitution of $(3.10)_2$ on the right-hand side of (B1), it follows at once that

$$(F_{iL}F_{iM})_{,K} = C_{LM,K} \quad . \tag{B2}$$

Thus, any solution F_{iL} to (3.10) satisfying the condition that $F_{iL}F_{iM} = C_{LM}$ at one point will have the property that $F_{iL}F_{IM} = C_{LM}$ everywhere. Moreover, since the right-hand side of (3.10) is symmetric with respect to the indices (K,M), it follows that a solution F_{iL} to (3.10) will also satisfy (3.6) and this, in turn, ensures the existence of the deformation function $\underset{\sim}{\chi}$.

It was remarked following (3.11) that since $\underset{\sim}{U}$ can be determined uniquely from the knowledge of $\underset{\sim}{C}$, with the use of (2.1), we obtain the differential equations (3.12) for the determination of the rotation $\underset{\sim}{R}$. To show this, we substitute (2.1) into the expression on the left-hand side of (3.10) and obtain

$$F_{iL}F_{iM,K} = R_{iN}U_{NL}(R_{iP}U_{PM})_{,K} = U_{NL}U_{NM,K} + U_{NL}R_{iN}R_{iP,K}U_{PM} \quad ,$$

or after rearrangement of terms

$$R_{iM}R_{iN,K} = U^{-1}_{ML}(F_{iL}F_{iP,K} - U_{QL}U_{QP,K})U^{-1}_{PN} \quad . \tag{B3}$$

Next, we substitute from (3.10) on the right-hand side of (B3), make use of $C_{KL} = U_{KM}U_{ML}$ by $(2.3)_2$ along with $\underset{\sim}{R}^T \underset{\sim}{R}_{,K} = (R_{iM}R_{iN,K})\underset{\sim}{e}_M \otimes \underset{\sim}{e}_N$ and obtain (3.14) and (3.15). The component forms of these equations, namely

$$\underset{\sim}{R}^T \underset{\sim}{R}_{,K} = \tfrac{1}{2}\{U^{-1}_{ML}(U_{NL,K} - U_{NK,L}) + (U_{MK,L} - U_{ML,K})U^{-1}_{LN} + U^{-1}_{MP}(U_{QP,L} - U_{QL,P})U_{QK}U^{-1}_{LN}\}\underset{\sim}{e}_M \otimes \underset{\sim}{e}_N \tag{B4}$$

correspond to Eqs. (8)-(10) in Shield's paper (1973).

To obtain the necessary and sufficient condition for the existence of $\underset{\sim}{R}$, we first recall that the necessary and sufficient condition for the existence of the deformation gradient tensor $\underset{\sim}{F}$ which satisfies (3.11) is that

$$\underset{\sim}{F}_{,KL} = \underset{\sim}{F}_{,LK} \quad \text{or} \quad F_{iM,KL} = F_{iM,LK} \ . \tag{B5}$$

Again substitute from (2.1), and after cancellation of identical terms, (B5) becomes

$$\underset{\sim}{R}_{,KL}\underset{\sim}{U} + \underset{\sim}{R}\,\underset{\sim}{U}_{,KL} = \underset{\sim}{R}_{,LK}\underset{\sim}{U} + \underset{\sim}{R}\,\underset{\sim}{U}_{,LK} \ . \tag{B6}$$

Clearly for a given $\underset{\sim}{U}$, $\underset{\sim}{U}_{,KL} = \underset{\sim}{U}_{,LK}$ and since $\underset{\sim}{U}$ is nonsingular, it follows from (B6) that the necessary and sufficient condition for the existence of $\underset{\sim}{R}$ is

$$\underset{\sim}{R}_{,KL} = \underset{\sim}{R}_{,LK} \ . \tag{B7}$$

Alternatively, we may obtain the conditions for the existence of $\underset{\sim}{R}$ by considering the required conditions for the existence of a solution of (3.14). Thus, from (3.14) we write $\underset{\sim}{R}_{,K} = \underset{\sim}{R}\,\underset{\sim}{A}_K$ and after substitution from (B5) or equivalently (B7) we obtain:

$$\underset{\sim}{A}_K\underset{\sim}{A}_L - \underset{\sim}{A}_L\underset{\sim}{A}_K + \underset{\sim}{A}_{L,K} - \underset{\sim}{A}_{K,L} = 0 \ . \tag{B8}$$

This corresponds to Eq. (11) in Shield's paper (1973), which was obtained as a special case of more general results discussed by Thomas (1934).

In order to verify the truth of the inequality (4.4), we first observe that for real numbers a and b, $0 < a < b$, the following two inequalities hold:

$$a^n < b^n \quad , \quad a^{\frac{1}{n}} < b^{\frac{1}{n}} \quad . \tag{B9}$$

Next, with the help of $(B9)_2$, from the result (4.3) which holds for k,ℓ positive integers, we obtain $(\varepsilon_1^k)^{1/k} < (\overline{C}\varepsilon_o^\ell)^{1/k}$ or $\varepsilon_1 < \overline{C}^{1/k}\varepsilon_o^{\ell/k}$ and hence

$$\varepsilon_1^n < (\overline{C}^{1/k}\varepsilon_o^{\ell/k})^n \quad . \tag{B10}$$

Then, from the condition on $\|h_1(\underset{\sim}{E}_{,K})\|$ given following (4.2), namely $\|h_1(\underset{\sim}{E}_{,K})\| < D\varepsilon_1^n$ as $\varepsilon_1 \to 0$, the inequality (4.4) follows from (B10).

To verify the truth of (4.19), with the use of (4.18) consider the scalar $\underset{\sim}{\alpha}_K \cdot \underset{\sim}{R}^T\underset{\sim}{R}_{,K}\underset{\sim}{\gamma}_K$ and observe the identities (no sum on K)

$$\begin{aligned}
&\underset{\sim}{\alpha}_K \cdot (\underset{\sim}{\mu} \otimes \underset{\sim}{\mu}_{,K})\underset{\sim}{\gamma}_K = \underset{\sim}{\mu} \cdot \underset{\sim}{\alpha}_K(\underset{\sim}{\mu}_{,K} \cdot \underset{\sim}{\gamma}_K) \quad , \\
&\underset{\sim}{\alpha}_K \cdot [(\underset{\sim}{\mu}_{,K} \times \underset{\sim}{\mu}) \otimes \underset{\sim}{\mu}]\underset{\sim}{\gamma}_K = \underset{\sim}{\mu} \cdot \underset{\sim}{\gamma}_K(\underset{\sim}{\mu}_{,K} \times \underset{\sim}{\mu} \cdot \underset{\sim}{\alpha}_K) \quad , \\
&\underset{\sim}{\alpha}_K \cdot \underset{\sim}{e}_L \otimes (\underset{\sim}{\mu}_{,K} \times \underset{\sim}{e}_L)\underset{\sim}{\gamma}_K = (\underset{\sim}{\alpha}_K \cdot \underset{\sim}{e}_L)(\underset{\sim}{\mu}_{,K} \times \underset{\sim}{e}_L \cdot \underset{\sim}{\gamma}_K) = \underset{\sim}{\mu}_{,K} \times (\underset{\sim}{\alpha}_K \cdot \underset{\sim}{e}_L)\underset{\sim}{e}_L \cdot \underset{\sim}{\gamma}_K \\
&\qquad = \underset{\sim}{\mu}_{,K} \times \underset{\sim}{\alpha}_K \cdot \underset{\sim}{\gamma}_K \quad , \\
&\underset{\sim}{\alpha}_K \cdot \underset{\sim}{e}_L \otimes (\underset{\sim}{\mu} \times \underset{\sim}{e}_L)\underset{\sim}{\gamma}_K = (\underset{\sim}{\alpha}_K \cdot \underset{\sim}{e}_L)(\underset{\sim}{\mu} \times \underset{\sim}{e}_L \cdot \underset{\sim}{\gamma}_K) = \underset{\sim}{\mu} \times \underset{\sim}{\alpha}_K \cdot \underset{\sim}{\gamma}_K \quad .
\end{aligned} \tag{B11}$$

Since $(\underset{\sim}{\mu},\underset{\sim}{\gamma}_K,\underset{\sim}{\alpha}_K)$ form a right-handed orthonormal triad, $(B11)_{1,2}$ vanish identically and $(B11)_4$ is equal to -1. In addition, since $\underset{\sim}{\alpha}_K$ is directed along $\underset{\sim}{\mu}_{,K}$, $(B11)_3$ vanishes also and the scalar $\underset{\sim}{\alpha}_K \cdot \underset{\sim}{R}^T\underset{\sim}{R}_{,K}\underset{\sim}{\gamma}_K$ reduces to (4.19)

We next establish the validity of the inequality (4.28), where θ is the angle of rotation for the tensor $\underset{\sim}{R}$, and α is the angle between a unit

vector $\underset{\sim}{v}$ and the rotated unit vector $\underset{\sim}{R}\,\underset{\sim}{v}$. With the help of the representation (2.13), we have

$$\cos\alpha = \underset{\sim}{v} \cdot (\underset{\sim}{R}\,\underset{\sim}{v}) = (\underset{\sim}{\mu} \cdot \underset{\sim}{v})^2 (1 - \cos\theta) + \cos\theta \quad . \tag{B12}$$

Observing that $(1 - \cos\theta) \geq 0$, and $(\underset{\sim}{\mu} \cdot \underset{\sim}{v})^2 \geq 0$, we conclude from (B12) that $\cos\alpha \geq \cos\theta$ and (4.28) is verified.

Appendix C

In a recent paper, Casey and Naghdi (1981) have shown that a theory in which both the deformation and deformation gradient are small can be constructed so as to possess desirable invariant properties in that the constitutive equations and the equations of motion are properly invariant under arbitrary (not necessarily infinitesimal) superposed rigid body motions. The method of construction is effectively such that they consider the rotation at one material point, called the pivot Y, and then remove from every point of the body this rotation obtaining a new configuration $\underset{\sim}{\kappa}^*$, and then show that in the configuration $\underset{\sim}{\kappa}^*$ all constitutive results are properly invariant. In the second part of their work, they demonstrate that to the order of approximation of the kinematical result, it does not matter which particle is chosen as pivot.

It is the purpose of the present appendix to show that the choice of pivot is immaterial also in the presence of moderate rotation or more specifically for (4.60), $(4.70)_1$, (4.73) and (4.75). Temporarily, we attach the subscripts Y and Y′ to quantities $\underset{\sim}{F}^*, \underset{\sim}{H}^*$, etc., when the pivots Y and Y′ are identified with material points whose position vectors are $\underset{\sim}{X} = {}_0\underset{\sim}{X}$, and $\underset{\sim}{X} = \underset{\sim}{X}' \neq {}_0\underset{\sim}{X}$, respectively, in the reference configuration $\underset{\sim}{\kappa}_0$. Using (4.60) and (4.51), the rotation tensor at $\underset{\sim}{X}'$ is given by

$$\underset{\sim}{R}_{Y'} = \underset{\sim}{R}(\underset{\sim}{X}',t) = \underset{\sim}{R}_0(\underset{\sim}{I} + \underset{\sim}{\Psi}' + \underset{\sim}{\Phi}') \quad , \tag{C1}$$

where

$$\underset{\sim}{\Phi}' = \underset{\sim}{\Phi}(\underset{\sim}{X}',t) = \underset{\sim}{0}(\varepsilon_0) \quad \text{as} \quad \varepsilon_0 \to 0 \quad ,$$
$$\underset{\sim}{\Psi}' = \underset{\sim}{\Psi}(\underset{\sim}{X}',t) = -\underset{\sim}{\Psi}'^T = \underset{\sim}{0}(\varepsilon_0^{1/2}) \quad , \quad \underset{\sim}{\Psi}'\underset{\sim}{\Psi}' = 2\underset{\sim}{\Phi}' \quad . \tag{C2}$$

Recall the form of the rotation tensor in (4.60) and multiply the right-hand side by $\underset{\sim}{R}_{Y'}\underset{\sim}{R}_{Y'}^T = \underset{\sim}{I}$. With the help of (C1), the tensor $\underset{\sim}{R}$ can be written as

$$\begin{aligned}
\underset{\sim}{R} &= (\underset{\sim}{R}_{Y'}\underset{\sim}{R}_{Y'}^{T})\underset{\sim}{R}_{o}(\underset{\sim}{I}+\underset{\sim}{\Phi}+\underset{\sim}{\Psi}) \\
&= \underset{\sim}{R}_{Y'}(\underset{\sim}{I}+\underset{\sim}{\Phi}'+\underset{\sim}{\Psi}')^{T}(\underset{\sim}{I}+\underset{\sim}{\Phi}+\underset{\sim}{\Psi}) \\
&= \underset{\sim}{R}_{Y'}(\underset{\sim}{I}+\underset{\sim}{\Phi}+\underset{\sim}{\Phi}'+\underset{\sim}{\Psi}-\underset{\sim}{\Psi}'-\underset{\sim}{\Psi}'\underset{\sim}{\Psi}) \\
&= \underset{\sim}{R}_{Y'}\{\underset{\sim}{I}+[\underset{\sim}{\Phi}+\underset{\sim}{\Phi}'-\tfrac{1}{2}(\underset{\sim}{\Psi}'\underset{\sim}{\Psi}+\underset{\sim}{\Psi}\,\underset{\sim}{\Psi}')] \\
&\qquad + [\underset{\sim}{\Psi}-\underset{\sim}{\Psi}'-\tfrac{1}{2}(\underset{\sim}{\Psi}'\underset{\sim}{\Psi}-\underset{\sim}{\Psi}\,\underset{\sim}{\Psi}')]\}
\end{aligned} \tag{C3}$$

as $\varepsilon_o \to 0$, where terms of $\underset{\sim}{0}(\varepsilon_o^{3/2})$ as $\varepsilon_o \to 0$ have been neglected. Consider the quantities

$$\begin{aligned}
\underset{\sim}{\overline{\Phi}} &= \underset{\sim}{\overline{\Phi}}(\underset{\sim}{X},t) = \underset{\sim}{\Phi}+\underset{\sim}{\Phi}'-\tfrac{1}{2}(\underset{\sim}{\Psi}'\underset{\sim}{\Psi}+\underset{\sim}{\Psi}\,\underset{\sim}{\Psi}') = \underset{\sim}{\overline{\Phi}}^{T} \quad , \\
\underset{\sim}{\overline{\Psi}} &= \underset{\sim}{\overline{\Psi}}(\underset{\sim}{X},t) = \underset{\sim}{\Psi}-\underset{\sim}{\Psi}'-\tfrac{1}{2}(\underset{\sim}{\Psi}'\underset{\sim}{\Psi}-\underset{\sim}{\Psi}\,\underset{\sim}{\Psi}') = -\underset{\sim}{\overline{\Psi}}^{T}
\end{aligned} \tag{C4}$$

and observe that $\underset{\sim}{\overline{\Phi}} = \underset{\sim}{0}(\varepsilon_o)$ and $\underset{\sim}{\overline{\Psi}} = \underset{\sim}{0}(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$. By direct calculation after neglecting terms of $\underset{\sim}{0}(\varepsilon_o^{3/2})$ as $\varepsilon_o \to 0$, we conclude that

$$\underset{\sim}{\overline{\Psi}}\,\underset{\sim}{\overline{\Psi}} = 2\underset{\sim}{\overline{\Phi}} \quad \text{as} \quad \varepsilon_o \to 0 \quad . \tag{C5}$$

In addition, both $\underset{\sim}{\overline{\Phi}}$ and $\underset{\sim}{\overline{\Psi}}$ vanish at $\underset{\sim}{X} = \underset{\sim}{X}'$. Thus, (C3) has the form

$$\begin{gathered}
\underset{\sim}{R} = \underset{\sim}{R}_{Y'}(\underset{\sim}{I}+\underset{\sim}{\overline{\Phi}}+\underset{\sim}{\overline{\Psi}}) \quad , \\
\underset{\sim}{\overline{\Phi}} = \underset{\sim}{\overline{\Phi}}^{T} = \underset{\sim}{0}(\varepsilon_o) \quad \text{as} \quad \varepsilon_o \to 0 \quad , \quad \underset{\sim}{\overline{\Psi}} = -\underset{\sim}{\overline{\Psi}}^{T} = \underset{\sim}{0}(\varepsilon_o^{1/2}) \quad , \quad \underset{\sim}{\overline{\Psi}}\,\underset{\sim}{\overline{\Psi}} = 2\underset{\sim}{\overline{\Phi}} \quad , \\
\underset{\sim}{R}_{Y'} = \underset{\sim}{R}(\underset{\sim}{X}',t) \quad , \quad \underset{\sim}{\overline{\Phi}}(\underset{\sim}{X}',t) = \underset{\sim}{\overline{\Psi}}(\underset{\sim}{X}',t) = \underset{\sim}{0} \quad .
\end{gathered} \tag{C6}$$

Hence, we conclude that the form (4.60) remains unchanged by the choice Y' as the pivot.

From (4.66) and with the help of (C1) and (C4), we calculate

$$\underset{\sim}{F}^{*}_{Y'} = \underset{\sim}{R}^{T}_{Y'}\,\underset{\sim}{F}$$

$$= (\underset{\sim}{I} + \underset{\sim}{\Phi}' + \underset{\sim}{\Psi}')^{T}\underset{\sim}{R}_{o}^{T}\underset{\sim}{R}_{o}(\underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Psi} + \underset{\sim}{E})$$

$$= (\underset{\sim}{I} + \underset{\sim}{\Phi} + \underset{\sim}{\Phi}' + \underset{\sim}{\Psi} - \underset{\sim}{\Psi}' - \underset{\sim}{\Psi}'\underset{\sim}{\Psi} + \underset{\sim}{E})$$

$$= \underset{\sim}{I} + \underset{\sim}{\overline{\Phi}} + \underset{\sim}{\overline{\Psi}} + \underset{\sim}{E} \quad , \tag{C7}$$

where terms of $\underset{\sim}{0}(\varepsilon_o^{3/2})$ as $\varepsilon_o \to 0$ have been neglected. Thus,

$$\underset{\sim}{H}^{*}_{Y'} = \underset{\sim}{F}^{*}_{Y'} - \underset{\sim}{I} = \underset{\sim}{\overline{\Phi}} + \underset{\sim}{\overline{\Psi}} + \underset{\sim}{E} \quad . \tag{C8}$$

Using (C8), we again have

$$\underset{\sim}{E}^{*}_{Y'} = \tfrac{1}{2}(\underset{\sim}{F}^{*T}_{Y'}\,\underset{\sim}{F}^{*}_{Y'} - \underset{\sim}{I})$$

$$= \tfrac{1}{2}(\underset{\sim}{H}^{*}_{Y'} + \underset{\sim}{H}^{*}_{Y'} + \underset{\sim}{H}^{*T}_{Y'}\,\underset{\sim}{H}^{*}_{Y'}) = \underset{\sim}{E} \quad . \tag{C9}$$

Hence, $(4.70)_1$, (4.73) and (4.75) are not affected by the choice of pivot.

In the above, we have $\underset{\sim}{E}_{,K} = \underset{\sim}{0}(\varepsilon_o^{1/2})$ as $\varepsilon_o \to 0$. It should be clear that similar procedures can be used for the case in which $\underset{\sim}{E}_{,K} = \underset{\sim}{0}(\varepsilon_o)$ as $\varepsilon_o \to 0$.

Before closing this appendix, we consider the consequence of the change of pivot Y on the stress tensor $\underset{\sim}{T}^{*}$ defined by (5.7). When a particle Y', different from Y, is chosen as pivot, $(5.8)_1$ becomes

$$\underset{\sim}{T}^{*}_{Y'} = \underset{\sim}{R}^{T}_{Y'}\,\underset{\sim}{T}\,\underset{\sim}{R}_{Y'} \quad , \tag{C10}$$

where $\underset{\sim}{R}_{Y'} = \underset{\sim}{R}\ (\underset{\sim}{X} = \underset{\sim}{X}', t)$ is a function of time only. Keeping in mind that $\underset{\sim}{\mathcal{K}}[\underset{\sim}{E}] = \underset{\sim}{0}(\varepsilon_o)$ as $\varepsilon_o \to 0$, with the help of (C1), (C2), (5.7) and (5.8), to the order of approximation considered we obtain

$$\underset{\sim}{T}^*_{Y'} = \rho_o(\underset{\sim}{I} + \underset{\sim}{\Phi}' + \underset{\sim}{\Psi}')\underset{\sim}{R}_o^T\underset{\sim}{R}_o \underset{\sim}{\mathcal{K}}[\underset{\sim}{E}]\underset{\sim}{R}_o^T\underset{\sim}{R}_o(\underset{\sim}{I} + \underset{\sim}{\Phi}' + \underset{\sim}{\Psi}')$$

$$= \rho_o \underset{\sim}{\mathcal{K}}[\underset{\sim}{E}] = \underset{\sim}{T}^*_Y \quad . \tag{C11}$$

where in writing $(C11)_2$ terms of $\underset{\sim}{0}(\varepsilon_o^{3/2})$ as $\varepsilon_o \to 0$ have been neglected. Hence, to within terms of $\underset{\sim}{0}(\varepsilon_o^{3/2})$, $\underset{\sim}{T}^*$ is unaffected by a change of pivot.

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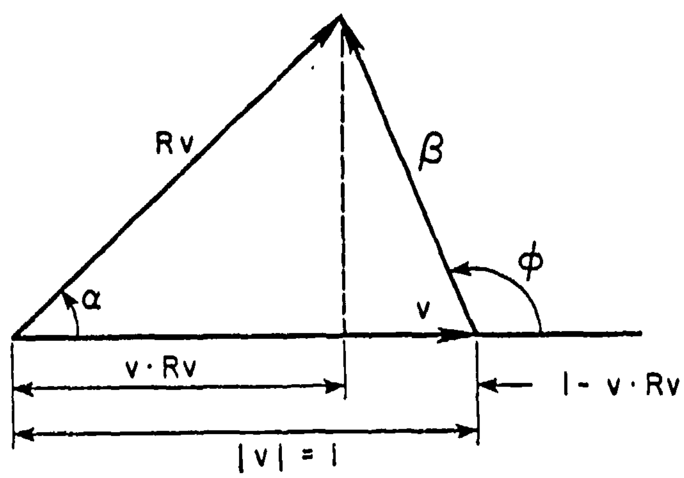

Fig. 1 A sketch showing the vector $\underset{\sim}{R}\,\underset{\sim}{v}$ obtained by rotating a unit vector $\underset{\sim}{v}$ through an angle α and the vector $\underset{\sim}{\beta}$ representing the difference of $\underset{\sim}{v}$ and the rotated vector $\underset{\sim}{R}\,\underset{\sim}{v}$.